Applied Mathematics IIB

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This lecture notes outline a precise definition and theorem as well as the examples of definition sometime also examples of theorem with a precise proofs. This lecture is about giving a definition, theorem and examples of sequence, series, power series, differential calculus of function of several variables and multiple integrals.

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Contents

Chapter one

Sequence and series

Definition of sequence

A sequence is a function whose domain is the set of positive integers. The numbers in the range of the sequence, which are called the elements of the sequence, are restricted to real numbers in this lesson.

Or A sequence is a list of numbers $a_{1,}a_2, a_3,....a_n,....$ in a given order. Each of $a_{1,}a_2, a_3,....a_n,....$ represents the number. These are the **terms** of the sequence.

For instance, $2_{,}4,6,....2n,....$ has the first term $a_1 = 2, a_2 = 4, a_3 = 6,....a_n = 2n,....$

The integer n is called the **index** of a_n , and indicates where a_n occurs in the list.

Definition of Infinite Sequence

An **infinite sequence** of numbers is a function whose domain is the set of positive integers.

The function associated to the sequence $2,4,6,8,10,...2n,...$ sends 1 to a_1 , 2 to a_2 and so on. The general behavior of this sequence is describedby the formula $a_n = 2n,$ where the initial index is 1 or simply written as $\{a_n = 2n\}_{n=1}$.

The sequence $12,14,16,18,20,22,....$ is described by the formula $a_n = 10 + 2n,$ where the initial index is 6 or simply written as $\{a_n = 2n\}_{n=6}$.

It can also be described by the simpler formula $a_n = 2n$ where the index *n* starts at 6 and increases.

Remark: The **order** of the sequence is very important.

Sequences can be described by writing rules or formulas that specify their terms.

A sequence can be described in the following three ways:-

1. Listing the elements of the sequence
2. Graph
3. General formula

For instance,

a) $a_n = \frac{1}{n}$

b) $a_n = \frac{1}{n(n+1)}$

c) $a_n = (-1)^n \frac{1}{n}$

d) $a_n = \sqrt{n}$

e) $a_n = \sqrt{1+\sqrt{n}}$

And the sequence can be described by listing the values of the sequence.

For instance to the above sequence we have

a) $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, ...\}$

b) $\{a_n\} = \{\frac{1}{2}, \frac{1}{3\times 4}, \frac{1}{4\times 5}, ..., \frac{1}{n(n+1)}, ...\}$

c) $\{a_n\} = \{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, ..., (-1)^n \frac{1}{n}, ...\}$

Activity -1

Write the terms of the above sequence d and e.

Note that the sequence sometimes denoted by the notation $\{a_n\}_{n=n_0}^{\infty}$, where n_0 the **initial index**.

For instance the sequence $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{n}, ...\}$ can be written as $\{a_n\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

Arithmetic sequence and Geometric sequence (progression)

Definition (Arithmetic sequence is a sequence on which the difference between consecutive terms is constant. This constant is called common difference and denoted by'd' this is to mean that if

$\{a_n\}$ be an arithmetic sequence with common difference d ,then w have $a_{n+1} - a_n = d, \forall n$

Theorem

If $\{a_n\}$ be an arithmetic sequence with first term a_1 and common difference d ,then the n^{th} term is given $a_n = a_1 + (n-1)d$

e.g

1. Find the general formula of the following arithmetic sequence
   a. $1,2,3,4,5,....$
   b. $1,6,11,16,21,....$
2. Given an arithmetic sequence with first term 5 and common difference 4.
   a. Find the first five terms of the sequence
   b. Find the twentieth term of the sequence

Definition (geometric sequence)

lhsfkwhwfw;wf lnlfnbjkjkeyevgjjfdrdmgg.jujiee nwfgqgqdnnwkglgu

mxf

Since a sequence is a function, then, in particular, the sequence $\{a_n\}_{n=n_0}^{\infty}$ can have the following **properties**

a) It is bounded (upper bound and lower bound) if there exists a constant $K > 0$ such that $|a_n| \le K$ for all n.

b) It is monotone increasing if $a_n \le a_{n+1}$ for all n, and it is monotone decreasing if $a_n \ge a_{n+1}$ for all n

c) It converges to a finite number c if $\lim_{n\to\infty} a_n = c$,that is, for a given $\in > 0$ there exists an integer N such that $|a_n - c| < \in$

if $n > N$. In this case c is called the limit of the sequence. If the limit of the sequence does not converge then the sequence is said to be divergent.

d) It is said to oscillate if it does not converges to a finite limit ,nor to $+\infty$ or $-\infty$ as $n \to \infty$.

Activity-2

1) Determine whether the following sequence converges or diverges.

a) $a_n = \dfrac{n^2 + 2n}{2n^2 + 5n}$

b) $a_n = \sqrt{n+1} - \sqrt{n}$

c) $a_n = \dfrac{2^n}{n^3}$

d) $a_n = (-1)^n$

e) $a_n = \dfrac{\ln n}{n}$

f) $a_n = \dfrac{n!}{n^n}$

g) $a_n = \tan^{-1} n$

h) $a_n = \dfrac{e^n + e^{-n}}{e^{2n} - 1}$

2) Determine if the following sequences are monotonic /or bounded.

a) $\{-n\}_{n=0}^{\infty}$

b) $\{(-1)^{n+1}\}_{n=1}^{\infty}$

c) $\left\{\dfrac{1}{n^2}\right\}_{n=1}^{\infty}$

3) Write the initial four terms of the following sequence

a) $a_n = \left\{\frac{1}{3}\right\}_{n=3}^{\infty}$

b) $a_n = \left\{\frac{k-1}{3n+1}\right\}_{n=3}^{\infty}, where k = cons\tan t$

c) $a_n = \left\{\frac{1}{n}\right\}_{n=3}^{\infty}$

4) Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

a) $\{2,7,12,17,....\}$

b) $\left\{1,\frac{-2}{3},\frac{4}{9},\frac{-8}{27},....\right\}$

c) $\{5,1,5,1,....\}$

Convergence properties of the sequence

Since the sequences are functions, we may add, subtract, multiply, and divide.Let $\{a_n\}_{n=m}^{\infty}, and \ \{b_n\}_{n=m}^{\infty}$ be convergent sequences. Then

a) $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

b) $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$, where c is a scalar.

c) $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$

d) $\lim_{n\to\infty}(\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$,provided that $\lim_{n\to\infty} b_n \neq 0$

e) $\lim_{n\to\infty} a_n{}^p = [\lim_{n\to\infty} a_n]^p \ provided \ a_n \geq 0$

Squeeze theorem for the sequence

If $a_n \leq c_n \leq b_n$ for all $n \geq M$ for some M and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ then $\lim_{n\to\infty} c_n = L$

Examples: - Use the ***Squeeze theorem*** to determine the convergence or divergence of the following sequence.

i. $a_n = \frac{\sin^2 n}{n}$

Solution: - by using the squeezing theorem we can show that whether the given sequence is convergent or divergent. Recall that the sine function is an element of $[-1,1]$

Or $\sin n \in [-1,1]$ but in case of the sequence the domain is restricted and range not at all it depends on the nature of the given sequence.

$\Rightarrow \sin n \in [-1,1]$

$\Rightarrow \sin^2 n \in [0,1], why?$

$\Rightarrow 0 \le \sin^2 n \le 1, defn.of, \text{int} erval$

$\Rightarrow \frac{0}{n} \le \frac{\sin^2 n}{n} \le \frac{1}{n}$ Bydividing all sides for n

$\Rightarrow \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\sin^2 n}{n} \le \lim_{n\to\infty} \frac{1}{n}$ By taking the limit to all sides

Clearly, as the limit of the end functions exist and equal to each other i.e zero

By using squeezing theorem we can deduce that $\lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$

Activity -3(find the limit of the following sequence)

ii. $a_n = \frac{\cos^2 n}{n}$

iii. $a_n = \frac{\sin n}{n}$

iv. $a_n = \frac{n!}{n^n}$

Theorem

If $\lim_{n\to\infty} |a_n| = 0, then \lim_{n\to\infty} a_n = 0.$

Proof: The main thing to this proof is to note that,

$-|a_n| \le a_n \le |a_n|$

$\Rightarrow -\lim_{n\to\infty}|a_n| \le \lim_{n\to\infty} a_n \le \lim_{n\to\infty}|a_n|$ Bytaking the limit $as n \to \infty$

By squeezing theorem we note that $\lim_{n\to\infty} a_n = 0$ as a required answer.

Definition (sequence and subsequence)

Notice that $\{a_n\}_{n=1}^{\infty}$ is the sequence $a_{1,} a_2, a_3, \ldots . a_n, \ldots .$ similarly,

$\{a_{n+1}\}_{n=1}^{\infty}$

is the sequence

$a_2, a_3, \ldots . a_n, \ldots .$

which is the same as $\{a_n\}_{n=2}^{\infty}$

Since $\{a_n\}_{n=1}^{\infty}$ and $\{a_n\}_{n=2}^{\infty}$ have the same terms except for a_1 in the first sequence, it follows that

$\{a_{n+1}\}_{n=1}^{\infty}$ haslimit if and only if $\{a_n\}_{n=1}^{\infty}$ does, and in that case $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$,

Note that:- a_{n+1} is called a **subsequence** of a_n

For instance, the sequence

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots ..$ And

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots ..$ has the same limit, namely zero.

Activity-4

Determine whether the sequence define d by $a_1 = 1$ and $a_{n+1} = 4 - a_n$ for $n \ge 1$ is

a) Convergent
b) Divergent

Theorem

If $\{a_n\}_{n=m}^{\infty}$ converges, then $\{a_n\}_{n=m}^{\infty}$ is bounded. (Proof exercise)

.

If $\{a_n\}_{n=m}^{\infty}$ is unbounded, then $\{a_n\}_{n=m}^{\infty}$ diverges.

Caution:the converse of the above theorem does not work.

Activity-4

Give Counter examples the above theorem.

Ans. $\{(-1)^n\}_{n=1}^{\infty}$

Theorem

A bounded sequence $\{a_n\}_{n=m}^{\infty}$ that is either increasing or decreasing converges.

Example

1. Given $a_n = \frac{n-1}{n}$,then apply this theorem and determine the convergence of this sequence.

Solution: - to do this we need to check the monotones and boundedness of the given sequence.

Monotones

Suppose the given sequence is increasing that is

$a_n \le a_{n+1}$

$\Rightarrow \frac{n-1}{n} \le ?\frac{n}{n+1}, where a_{n+1} = \frac{n}{n+1}$

$\Rightarrow n^2 - 1 \le ?n^2$, by taking cress-cross

$\Rightarrow n^2 - n^2 - 1 \le ?n^2 - n^2,$

$\Rightarrow -1 \le ?0$ Which is definitely true?

So, thesequence is increasing or $a_n = \frac{n-1}{n} i \sin crea \sin g(\uparrow)$

Now we are left with checking the boundedness

Boundedness

The sequence is clearly bounded below by zero and above by one, that is,

$\left[0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ... \to 1\right]$ by substituting the values of n starting from 1.

Hence, from the above theorem we see that the sequence converges.

2. investigate the convergence of the sequence $\left\{\frac{2^n}{n!}\right\}_{n=1}^{\infty}$

Solution

First, note that we do not know how to compute $\lim_{n\to\infty} \frac{2^n}{n!}$ .This has theindeterminate

form $\frac{\infty}{\infty}$

, but we cannot use L'Hopital's Rule here directly or indirectly. (Why not?)
So, the only option we do have is to apply the above theorem to determine the convergence and divergence of the sequence. Hence let we check monotone and boundedness respectively.

Monotone

By using eventually principle we have $a_n = \frac{2^n}{n!}, a_{n+1} = \frac{2^{n+1}}{(n+1)!}$. Suppose the sequence is

increasing such that $a_n \le a_{n+1} \Rightarrow \frac{2^n}{n!} \le \frac{2^{n+1}}{(n+1)!}$.

Now we need to check this if it is true we can say the sequence is increasing otherwise decreasing.

Hence,

$$a_n \le ?\, a_{n+1} \Rightarrow \frac{2^n}{n!} \le ?\, \frac{2^{n+1}}{(n+1)!}$$
$$\Rightarrow (n+1)!2^n \le n!2^{n+1}$$

$$\boxed{n! = 1\times 2\times 3\times 4\times --- \times n, 0! = 1}$$

$(n+1) \le ?2$ This is false if $n > 2$. Thus it is **decreasing sequence**.

Boundedness

Since the sequence is decreasing, we have that

$$\left|a_n\right| = \frac{2^n}{n!} \le \frac{2^1}{1!} = 2, \forall n \ge 1(bounded, by, 2)$$

Since the sequence is both **bounded** and **decreasin**g; we can say that the sequence is **convergent**.

3. Let $\{a_n\}_{n=0}^{\infty}$ be the sequence defined by $a_0 = 0 \; \& \; a_{n+1} = a_n{}^2 + \frac{1}{4}, \forall n \ge 0$ Then

   a. Determine the convergence or divergence of $\{a_n\}_{n=0}^{\infty}$

   b. If it converges find the limit

Solution:- we know that a bounded sequence that either increasing or decreasing is convergent. Hence, it is suffice to show that monotones and boundedness of the given sequence.

By definition of bounded sequence we know that $\left|a_n\right| \le K, \forall n \ge k$

In our case, $\left|a_n\right| \le \frac{1}{2} \, for \;\; n \ge 0.$ because $a_0 = 0$

$$\Rightarrow \left|a_0\right| < \frac{1}{2}$$

More over if $\left|a_n\right| < \frac{1}{2}.then$

$$\left|a_{n+1}\right| = \left|a_n{}^2 + \frac{1}{4}\right|$$

` $$\Rightarrow = a_n{}^2 + \frac{1}{4}$$

$$< (1/2)^2 + \frac{1}{4}$$

$$\left|a_{n+1}\right| < 1/2$$

Finally, we deduce that by using mathematical induction we can say that the sequence a_n is bounded.

Now, we are left with to show the monotones

To do this; let us assume that the sequence is increasing, that is, $a_n \le a_{n+1}$.

$$\Rightarrow a_n < a_{n+1}$$

$\Rightarrow a_n - a_{n+1} < 0$

$\Rightarrow a_n - a_n{}^2 + \frac{1}{4} < 0$

$\Rightarrow - a_n + a_n{}^2 - \frac{1}{4} > 0$

$\Rightarrow \left(a_n - \frac{1}{2}\right)^2 > 0$ This is valid if $a_n \neq \frac{1}{2}$ and holds for all other values.

Consequently the given sequence is increasing.

Hence, the sequence is convergent.

Since as it is convergent clearly it has a limit says it $L, i.e \lim_{n\to\infty} a_n = L.$

Aim: we need to find L

So, by definition of subsequence we have

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_n{}^2 + \frac{1}{4}) = L$ ....from basic properties of the limits.

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (a_n{}^2) + \frac{1}{4} = L$

$\Rightarrow (\lim_{n\to\infty} a_n)^2 + \frac{1}{4} = L$ ....from section above

$\Rightarrow (L)^2 + \frac{1}{4} = L$

$\Rightarrow L^2 - L + \frac{1}{4} = 0$ By using quadratic equation we have $L = \dfrac{1 \pm \sqrt{1 - 4(1)(\frac{1}{4})}}{2}$

$L = \frac{1}{2}$

Therefore, $\lim_{n\to\infty} a_n = \frac{1}{2}$

Activity -5

1) Consider the sequence $\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}$,..... Then

a) write a recursion formula for $a_n forn \geq 2$

b) Find the limit of the sequence as n tends to positive infinite.

2) Consider the sequence $a_1 = \sqrt{k}, a_{n+1} = \sqrt{k + a_n}, andk > 0$ Then

a) write a recursion formula for $a_n forn \geq 2$

b) Find the limit of the sequence as n tends to positive infinite.

3) Determine the monotone of the sequence $b_n = \dfrac{n!}{e^n}$

Definition

For any integer *n* ≥ 1, the **factorial** *n*! is defined as the product of the first *n* positiveintegers, where $n! = 1.2.3....n \,\&\, 0! = 1$

Infinite series

Definition

The summation of the terms of the infinite sequence is called infinite series. Let $\{a_n\}_{n=1}^{\infty}$ be the infinite sequence and the expression $a_1 + a_2 + a_3 + ... + a_n + ...$ stands for the infinite series and shortly written as $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ and recall that $\sum$ is used to denote the summation and has a variety of names. The most common names are:-

¬Series notation

¬Summation notation

¬Sigma notation

Activity-6

Does it make sense to talk about the sum of infinitely many terms?

Remark: - it would be impossible to find a finite sum for the series.

For instance, consider the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ..,, + \frac{1}{2^n} + ...$

Now let we start adding the terms to get the cumulative sums such as $\frac{1}{2},\frac{3}{4},\frac{7}{8},.....$ and after the n^{th} term, we get $1-\frac{1}{2^n}$ which becomes one as n increases. So it is reasonable to say that the sum of this infinite series is one and we write $\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+......=1$

Now let we start adding the terms to

$example-2, consider \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+..,+\frac{1}{2^n}+....$ get the cumulative sums such as $1,3,6,10,15,......$ and after the n^{th} term, we get $\frac{n(n+1)}{2}$ which becomes very large as n increases.

Note that from example one and two above the series can be converges or diverges. The above cumulative sum is called the partial sums of the given series and can be calculated as follows:-

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 = s_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + ... + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

These partial sums form a new sequence $\{s_n\}$ which may or may not have a limit. If the limit exists we say that the series converges, and otherwise diverges.

N.B the convergence and divergence of the series can be determined by using the limit on the n^{th} partial sums.

Caution: Do not take directly the limit to the series to determine the convergence or divergence, it is illegal.

One of example of infinite series is Geometric series.

Geometric series

Definition

If each term of the series is obtained from the preceding one by multiplying it by the common ratio **r** then it is said to be **Geometric series**. The possibility of common ratio **r** of geometric series of the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1} + ..., a \neq 0.$$

is:-

$r = 1, r \neq 1, r > 1, r < 1, -1 < r < 1$

Let us consider each possibility of r separately as follows.

a) If **r=1**,then

$$s_n = a + a + a + ... + a + ... = na.$$

$$\Rightarrow \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} na = \pm\infty$$

As the limit does not exist then the Geometric series diverges in this case.

b) If $r \neq 1,$ then we have

$$s_n = a + ar + ar^2 + ... + ar^{n-1} \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(1)$$
$$rs_n = ar + ar^2 + ar^3 + ... + ar^{n-1} + ar^n \text{\_\_\_\_\_\_\_\_\_\_}(2)$$

By subtracting equation two from equation one we will obtain

$$s_n - rs_n = a - ar^n$$

$$\Rightarrow \quad s_n = \frac{a(1-r^n)}{1-r}. \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(3)$$

To determine the limit of equation 3 above we need to consider the value of **r.**

Case 1

If -1<r<1, then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

Thus, when

$|r| < 1 \Rightarrow -1 < r < 1$ The geometric series is convergent and its sum is:-

$\frac{a}{1-r}$

Case 2

If $r \le -1$ **or** $r > 1$ **, then**

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} = \pm\infty$$

Thus the Geometric series is divergent in this case.

Generally, the Geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ... + ar^{n-1} + ..., \quad a \neq 0, is :-$$

This geometric series converges when

$|r| < 1$

And diverges when

$|r| \ge 1$

Example-1

1. Use the knowledge of Geometric series and determine the convergence and divergence of the following series, and if the series is convergent be in mind that the sum is the value you obtained.

   a) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + ---$

   b) $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

   c) $\sum_{n=0}^{\infty} x^n, |x| < 1$

d) $\sum_{n=0}^{\infty} \frac{12}{(-5)^n}$

2. Examples of Geometric series and their sums

a. $\pi - e + \frac{e^2}{\pi} - \frac{e^3}{\pi^2} + \frac{e^4}{\pi^3} - .... = \sum_{n=1}^{\infty} \pi \left(\frac{-e}{\pi} \right)^{n-1}, here, a = \pi, r = \frac{-e}{\pi}, sum = \frac{\pi^2}{\pi + e}$

This implies the series converges and converges to the above sum.

b. $1 - 1 + 1 - 1 + ... = \sum_{n=1}^{\infty} (-1)^{n-1}$ the series diverges since $r = -1$

2. Find the rational number represented by the repeated decimal number

$0.784784784....$

Solution we can write

$0.784784784.... = 0.784 + 0.000784 + 0.000000784 + ...$

By dividing the second by the first, the third by the second, and so on of the terms of the sequence we find a common ratio

$r = 0.001 \,\&\, a = 0.784$

So since the given decimal is the sum of Geometric series and r<1, we have the sum

$\frac{a}{1-r} = \frac{0.784}{1-0.001} = \frac{784}{999}$

3. Show that whether the series

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Is convergent or divergent. If it converges find its sum.

Solution:-this is not a Geometric series ,so we go back to the definition of the partial sum and compute the n[th]partial sums as follows:-

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} (\frac{1}{i} - \frac{1}{i+1})$$

…..By using partial fraction decomposition

$$s_n=\sum_{i=1}^{n}\frac{1}{i(i+1)}=\frac{1}{1(1+1)}+\frac{1}{2.3}+\frac{1}{3.4}+----=\sum_{i=1}^{\infty}(\frac{1}{i}-\frac{1}{i+1})$$

$$\Rightarrow\sum_{i=1}^{n}(\frac{1}{i}-\frac{1}{i+1})=1-\frac{1}{2}+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+.....+(\frac{1}{n}-\frac{1}{n+1})$$

$$\Rightarrow s_n=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

$$\Rightarrow\sum_{i=1}^{n}(\frac{1}{i}-\frac{1}{i+1})=1-\frac{1}{2}'+(\frac{1}{2}'-\frac{1}{3}')+(\frac{1}{3}'-\frac{1}{4}')+.....+(\frac{1}{n}'-\frac{1}{n+1})$$

$$\Rightarrow s_n=1-\frac{1}{n+1}$$

Thus,

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\lim_{n\to\infty}(1-\frac{1}{n+1})=1$$

Harmonic series

Show that the harmonic series

$\sum_{n=1}^{\infty}\frac{1}{n}$ Is divergent, i.e, $\lim_{n\to\infty}s_n=\infty$

Solution: - we need to develop the nth partial sums of the series.

Now, consider

$$s_1 = 1$$
$$s_2 = 1 + \frac{1}{2}$$
$$s_3 = 1 + \frac{1}{2} + \frac{1}{3}$$
$$s_4 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) > 1 + \frac{2}{2} = \frac{2}{2}$$
$$\vdots$$
$$s_8 = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) = 1 + \frac{3}{2}$$
$$\vdots$$
$$s_{16} = 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + (\frac{1}{9} + ... + \frac{1}{15} + \frac{1}{16}) = 1 + \frac{4}{2}$$
$$s_{2^n} > 1 + \frac{n}{2}$$

Now by taking the limit we obtain,

$\lim_{n\to\infty} s_{2^n} > \lim_{n\to\infty} (1 + \frac{n}{2})$, Which implies that the limit does not exist.

Thus, the ***harmonic series*** diverges.

4. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and find its sum.

Solution

Here we are given with the series is convengent. And we need to show that it is convergent. We know that the partial decomposition of the rational function

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{1}{n} - \frac{1}{n+1}$$

By taking the summation notation to both sides of the above equality we see that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + ...\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1} = s_n$$

This is the partial sums of the series.

Therefore $\lim_{x\to\infty} s_n = 1 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

This is an example of a **telescoping series,** so called because the partial sums fold up into a simple form when the terms are expanded in partial fractions. The sum is called telescoping sum **(collapsing sum)**.

Theorem

If $p \in z^+$, then $\sum_{n=1}^{\infty} \frac{1}{n(n+p)} = \frac{1}{p}\sum_{n=1}^{p} \frac{1}{p}$

Corollary

If $p \in z^+$ and c is a constant, then $\sum_{n=1}^{\infty} \frac{c}{n(n+p)} = \frac{c}{p}\sum_{n=1}^{p} \frac{1}{p}$

Theorem

If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{x\to\infty} a_n = 0$.

Proof

Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges say to L. So this means the sequence of the partial sums say S_k of the series defined by $S_k = \sum_{k=1}^{n} a_k$ also converges to L.

So,we have

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3 = S_2 + a_3$$

.

.

.

$$S_k = S_{k-1} + a_k = \sum_{n=1}^{k-1} a_n + a_k$$

Hence,

$\lim_{k\to\infty} a_k = \lim_{k\to\infty}(S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = L - L = 0$ as desired.

When considering whether a given series converges, the first question you should ask yourself is: "Does the n^{th} term approach 0 as n approaches ∞?" If the answer is no, then the series does not converge. If the answer is yes, then the series may or may not converge. If the sequence of terms $\{a_n\}$ tends to a nonzero limit L, then $\sum_{n=1}^{\infty} a_n$ diverges to infinity if $L > 0$ and diverges to negative infinity if $L < 0$.

Example

a. $\sum_{n=1}^{\infty} \frac{n}{2n-1}$ diverges to infinity since $\lim_{x\to\infty} \frac{n}{2n-1} = \frac{1}{2} > 0$

b. $\sum_{n=1}^{\infty} (-1)^n n \sin\left(\frac{1}{n}\right)$ diverges to infinity since $\lim_{x\to\infty} \left|(-1)^n n \sin\left(\frac{1}{n}\right)\right| = 1 > 0$

Theorem (test for divergence or nth term for divergence)

(a) If the $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(b) If $\lim_{n\to\infty} a_n = 0$, then the series $\sum_{n=1}^{\infty} a_n$ may either converges or diverges. (**proof coming soon)**

(1) Use the theorem and determine whether the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$

Theorem (properties of convergent series)

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converges to A and B respectively, then the following holds

a. $\sum_{n=1}^{\infty} ca_n$ converges to cA (for any c scalar number)

b. $\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges to $A \pm B$

c. If $a_n \le b_n$ for all $n = 1,2,3....,$ then $A \le B$

Example

a. Find the sum of the series $\sum_{n=1}^{\infty}\frac{1+2^{n+1}}{3^n}$

Solution

The given series is the sum of two geometric series,

$$\sum_{n=1}^{\infty}\frac{1}{3^n} = \sum_{n=1}^{\infty}\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1/3}{1-1/3} = \frac{1}{2} \ \&$$

$$\sum_{n=1}^{\infty}\frac{2^{n+1}}{3^n} = \sum_{n=1}^{\infty}\frac{2.2.2^{n+1}}{3.3^{n-1}} = \sum_{n=1}^{\infty}\frac{4.2^{n+1}}{3.3^{n-1}} = \sum_{n=1}^{\infty}\frac{4}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{4/3}{1-2/3} = 4$$

Thus its sum is $\frac{1}{2}+4=\frac{9}{2}$ by the above theorem

b. When dropped, an elastic ball bounces back up to a height three-quarter of that from which it fell. If the ball is dropped from a height of 2 m and allowed to bounce up and down indefinitely, what is the total distance it travels before coming to rest?

c. Determine whether the following series converges or diverges if it converges find its sum. And also determine the kind of the series.

   i. $\sum_{n=5}^{\infty}\frac{1}{(2+\pi)^{2n}}$

   ii. $\sum_{n=2}^{\infty}\frac{(-5)^n}{8^{2n}}$

   iii. $\sum_{k=0}^{\infty}\frac{2^{k+3}}{e^{k-3}}$

   iv. $\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$

v. $\sum_{n=1}^{\infty}\frac{1}{(3n-2)(3n+1)}$

Convergence tests

a. Convergence test for positive series

In the previous section we saw a few examples of convergent series (geometric and telescoping series) whose sums could be determined exactly because the partial sums s_n could be expressed in closed form as explicit functions of n whose limits as $n \to \infty$ could be evaluated. It is not usually possible to do this with a given series, and therefore it is not usually possible to determine the sum of the series exactly. However, there are many techniques for determining whether a given series converges and, if it does, for approximating the sum to any desired degree of accuracy.

In this section we deal exclusively with positive series, that is, series of the form

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$where $a_n \geq 0, \forall n \geq 1$

Theorem

An infinite series of positive terms is convergent if and only if its sequence of partial sums has an upper bound.

Example

Illustration of the above theorem by an example: show that the series $\sum_{n=1}^{\infty}\frac{1}{n!}$ is convergent.

Solution

We must find the upper bound for the partial sums of the series $\sum_{n=1}^{\infty}\frac{1}{n!}$

$$s_1 = 1, s_2 = 1+\frac{1}{2!}, s_3 = 1+\frac{1}{2!}+\frac{1}{1.2.3}, s_4 = 1+\frac{1}{2!}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}, s_n = 1+\frac{1}{2!}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+...+\frac{1}{1.2.3...n}$$

Now consider the first n terms of the **Geometric series** with $a=1$ and $r = 1/2$

$$\sum_{k=1}^{n}\frac{1}{2^{k-1}} = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}}$$

This geometric series has the sum of $\frac{a}{1-r}=2$ and we observe that the sum of each partial sum in the first calculation is less than 2. Hence we note that $\frac{1}{k!}\leq\frac{1}{2^{k-1}}$ and we have $s_n=\sum_{k=1}^{n}\frac{1}{k!}\leq\sum_{k=1}^{n}\frac{1}{2^{k-1}}<2$

From the above we see that s_n has an upper bound of 2. Therefore, by the above theorem the given series is convergent.

(1) The Integral Test

The integral test provides a means for determining whether an ultimately positive series converges or diverges by comparing it with an improper integral that behaves similarly.

Theorem (Integral Test)

Suppose that $f(n)=a_n$ where $f(n)$ is positive, continuous, and non-increasing on an $[N,\infty]$for some positive integer N .Then $\sum_{n=1}^{\infty}a_n$ & $\int_{N}^{\infty}f(t)dt$ either both converge or both diverge to infinity.

Proof (coming soon)

Remark If $f(n)=a_n$, where $f(n)$ is positive, continuous, and non-increasing on $[1,\infty]$, then the above Theorem assures us that $\sum_{n=1}^{\infty}a_n$ & $\int_{1}^{\infty}f(x)dx$ both converge or both diverge to infinity. It does not tell us that the sum of the series is equal to the value of the integral. The two are not likely to be equal in the case of convergence. However, as we see below, integrals can help us approximate the sum of a series.

(2) P-series

The principal use of the integral test is to establish the result of the following example concerning the series $\sum_{n=1}^{\infty}n^{-p}$ which is called a **p-series**. This result should be memorized; we will frequently compare the behavior of other series with **p-series** later in this and subsequent sections.

Example

a. Show that the p-series $\sum_{n=1}^{\infty} n^{-p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \{conveges, if, p > 1, diverges, if, p \leq 1$

Solution

In this proof we need to use integral test such that the condition of it satisfied. We see that if $p > 0$,then $f(x) = \frac{1}{x^p}$ is positive, continuous, and non-increasing on $[1, \infty]$.

In this space, we need to observe the following cases.

Case 1

If the value of p=1,the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$ which is called the harmonic series which is commonly diverges series. Or by using by integral test we have

$f(x) = \frac{1}{x}$,which is positive, continuous, and non-increasing on $[1, \infty]$

Hence, the improper integral becomes

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x} dx = \lim_{b \to \infty} \ln|x| \Big|_1^b = \lim_{b \to \infty} \ln|b| - \ln|1| = \infty$$

This implies the improper integral diverges, whichleads to the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Case 2

If the value of p oscillates between 0 and 1,i.e $0 < p < 1$,then the series becomes by using Integral test

$$\int_1^{\infty} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^p} dx = \lim_{b \to \infty} \int_1^{b} x^{-p} dx = \lim_{b \to \infty} \frac{x^{-p+1}}{1-p} \Big|_{1=x}^{b} = \lim_{b \to \infty} \left(\frac{b^{-p+1}}{1-p} - \frac{1^{-p+1}}{1-p} \right)$$

$$= \lim_{b \to \infty} \left(\frac{1}{(1-p)b^{p-1}} - \frac{1}{1-p} \right)$$

Note in the above manipulation the limit hugely depend on the value of p.

Subcase2

(a) If the value of $p < 1$,then the above limit becomes

$\lim_{b\to\infty}\left(\frac{1}{(1-p)b^{p-1}}-\frac{1}{1-p}\right)=\infty$ which means the improper integral diverges. Which leads to the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges if $p < 1$.

(b) If the value of $p > 1$,then the above limit becomes

$\lim_{b\to\infty}\left(\frac{1}{(1-p)b^{p-1}}-\frac{1}{1-p}\right)=\frac{1}{p-1}$ which means the improper integral converges and which leads to the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p > 1$.

Case3

If the value of p=0,the series becomes $\sum_{n=1}^{\infty}1$. By using the preceding theorem we see that $\lim_{n\to\infty} 1 = 1 > 0$ which means the series diverges to positive infinity.

In general the **p-series** of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$

a. Converges if $p > 1$
b. Diverges if $0 \le p \le 1$

(3) **Comparison Tests**

The next test we consider for positive series is analogous to the comparison theorem for improper integrals. It enables us to determine the convergence or divergence of one series by comparing it with another series that is known to converge or diverge.

Theorem (Comparison tests)

Let $\{a_n\}$ & $\{b_n\}$ be sequences for which there exists a positive constant K such that, ultimately,

$0 \le a_n \le Kb_n$

(a) If the series $\sum_{n=1}^{\infty} b_n$ converges, then so does the series $\sum_{n=1}^{\infty} a_n$.

(b) If the series $\sum_{n=1}^{\infty} a_n$ diverges to infinity, then so does the series $\sum_{n=1}^{\infty} b_n$

Proof.

Since a series converges if and only if its tail converges, we can assume, without loss of generality, that the condition $0 \le a_n \le Kb_n$ holds for all $n \ge 1$. let the partial sums are given by $s_n = a_1 + a_2 + a_3 + ...$ and $S_n = b_1 + b_2 + b_3 + b_4 + ...$ Then $s_n \le KS_n$. If the series $\sum_{n=1}^{\infty} b_n$,then $\{S_n\}$ is convergent and hence bounded. And hence $\{s_n\}$ is bounded by above. By preceding theorem the series $\sum_{n=1}^{\infty} a_n$ converges.Since the convergence of $\sum_{n=1}^{\infty} b_n$ guarantees that of $\sum_{n=1}^{\infty} a_n$ if the latter series diverges to infinity, then the former cannot converge either, so it must diverge to infinity too.

Example

Which of the following series converge? Give reasons for your answers.

(a) $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

(b) $\sum_{n=1}^{\infty} \frac{2n}{3n^3 - 1}$

(c) $\sum_{n=1}^{\infty} \frac{3n + 1}{n^3 + 1}$

(d) $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

(e) $\sum_{n=3}^{\infty} \frac{\log n}{n}$

Solution

In each case we must find a suitable comparison series that we already know converges or diverges.

(a) In this case we see that $0 < a_n < b_n = 0 < \frac{1}{2^n + 1} < \frac{1}{2^n}$ for $n = 1,2,3,....$ and since the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ convergent geometric series, then this leads to the series $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$ converges by comparison test.

(b) In this case again we need to search off the series we knew such that it either converges or diverges from the given series $\sum_{n=1}^{\infty} \frac{2n}{3n^3 - 1}$.

For $n \ge 1, n^3 \ge 1, so, 3n^3 - 1 \ge 2n^3$. Thus $a_n = \frac{2n}{3n^3 - 1} \le \frac{2n}{2n^3} = \frac{1}{n^2} = b_n$

But we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series principle that is

$p = 2 > 1$

Therefore the original series $\sum_{n=1}^{\infty} \frac{2n}{3n^3 - 1}$ converges by comparison test.

(c) Observe that $\frac{3n+1}{n^3+1}$ behaves like $\frac{3n}{n^3} = \frac{3}{n^2}$ for large n.So we would expect to compare the series with the convergent p-series $\sum_{n=1}^{\infty} \frac{3}{n^2}$. we have for $n \ge 1$,

$$a_n = \frac{3n+1}{n^3+1} = \frac{3n}{n^3+1} + \frac{1}{n^3+1} < \frac{3n}{n^3} + \frac{1}{n^3} < \frac{3}{n^2} + \frac{1}{n^2} = \frac{4}{n^2} = b_n .$$

Since the series $\sum_{n=1}^{\infty} \frac{4}{n^2}$ converges by **p-series** we can deduce also that the series $\sum_{n=1}^{\infty} \frac{3n+1}{n^3+1}$ converges by comparison test.

(d) For $n \ge 2$,we have $0 < \ln n < n$. Thus $a_n = \frac{1}{\ln n} > \frac{1}{n} = b_n$. Since the series $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges to infinity (harmonic series),so does the series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ by comparison test.

(e) Note that if $n \ge 3$, then $\log n > 1$. We can thus use the ''eventually'' form ofthe comparison test; we have $a_n = \frac{\log n}{n} > \frac{1}{n} = b_n$

Since the series $\sum_{n=3}^{\infty}\frac{1}{n}$ diverges to infinity (harmonic series),so does the series $\sum_{n=3}^{\infty}\frac{\log n}{n}$ by comparison test.

(4) **limit comparison test**

Theorem (limit comparison test)

Suppose that $\{a_n\}$ & $\{b_n\}$ are positive sequences and that $\lim_{n\to\infty}\frac{a_n}{b_n}=L$ where L is either a non-negative finite number or ∞.

(i) If $L<\infty$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ also converges.

(ii) If $L>0$ and $\sum_{n=1}^{\infty}b_n$ diverges, then $\sum_{n=1}^{\infty}a_n$ also diverges.

(iii) If $L=0$ and $\sum_{n=1}^{\infty}b_n$ converges, then $\sum_{n=1}^{\infty}a_n$ also converges.

Remark

In order to search for the value of another series for the corresponding given series is by considering the ratio of the highest degree that find in both denominator and numerator by disregarding the other left degrees.

Proof(exercise)

Example

1. Use the limit comparison test to determine whether the given series converges and diverges by showing all necessary step(s)

   a. $\sum_{n=1}^{\infty}\frac{n+5}{n^3-2n+3}$

   b. $\sum_{n=1}^{\infty}\frac{2n^2+3n}{\sqrt{n^5+5}}$

   c. $\sum_{n=1}^{\infty}\frac{1}{1+\sqrt{n}}$

   d. $\sum_{n=1}^{\infty}\frac{1+\sin n}{n^2}$

Solution

In all cases we need to search another series which is appropriate for comparison series by applying the above remark.

a. The dominant part of the numerator is n and the dominant part of the denominator is n^3.

Hence, we can let $a_n = \frac{n+5}{n^3-2n+3}, b_n = \frac{1}{n^2}$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n+5}{n^3-2n+3}}{\frac{1}{n^2}} = 1 > 0$

And we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as it is the p-series with $p > 1$

Since $L = 1 < \infty$ the series $\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ converges by limit comparison test.

b. The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5}$. The ratio these value is $b_n = \frac{2n^2}{\sqrt{n^5}}, a_n = \frac{2n^2+3n}{\sqrt{n^5+5}}$

Thus $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n^2+3n}{\sqrt{n^5+5}}}{\frac{2n^2}{\sqrt{n^5}}} = 1 > 0$ and we see that

$\sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{2n^2}{n^{5/2}} = \sum_{n=1}^{\infty} \frac{2}{n^{1/2}}, p-series, with, p = 0.5 < 1$. Since $\sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$ is divergent (p-series with $p < 1$)

(5) Alternative series test

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series,* whose terms alternatein sign. An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is called harmonic alternative series

b. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^n}$ is called geometric alternative series

Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - ...(b_n > 0)$ satisfy the condition

(a) $b_n b_{n+1} < 0$

(b) $b_{n+1} < b_n, decreasing, \forall n$

(c) $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

Proof (coming soon)

Example

Use the alternative series test and determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$

b. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3n}{4n-1}$

c. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2}{n^3+1}$

d. $\sum_{n=2}^{\infty}\frac{\cos \pi n}{\ln n}$

(6) Ratio Test

The comparison test and the limit comparison test hinge on first making a guess about convergence and then finding an appropriate series for comparison, both of which can be difficult tasks. But in this section and the next subsection we need to guess a series of comparison with the original series to determine the convergence and divergence of the series. And we will try to do a couple of the examples with a precise example. Enjoy it.

Condition of ratio test

Theorem (Ratio Test)

Suppose that $a_n > 0(ultimately)$ and that $\rho = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}$ exists or ∞.

a. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges

b. If $1 < \rho < \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c. If $\rho = 1$,then this test gives no information; the series may either converge or diverge to infinity.

Example

Test the following series for convergence:

(a) $\sum_{n=1}^{\infty} \frac{n^5}{2^n}$

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

(c) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

(d) $\sum_{n=1}^{\infty} \frac{2^n (n!)^2}{(2n)!}$

Solution

Since in all cases we need use Ratio test we should compute $\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

(7) Root test

In cases where it is difficult or inconvenient to find the limit required for the ratio test, this test is sometimes useful.

Theorem (Root Test)

Suppose that $a_n > 0(ultimately)$ and that $\delta = \lim_{n\to\infty} \sqrt[n]{a_n}$ exists or ∞.

a. If $0 \le \delta < 1$,then $\sum_{n=1}^{\infty} a_n$ converges

b. If $1 < \delta < \infty$, then $\lim_{n\to\infty} a_n = \infty$ and $\sum_{n=1}^{\infty} a_n$ diverges to infinity.

c. If $\delta = 1$, then this test gives no information; the series may either converge or diverge to infinity.

Examples

Test the following series for convergence:

a. $\sum_{n=1}^{\infty} \left(\frac{4n-5}{2n+3}\right)^n$

b. $\sum_{n=1}^{\infty} \frac{n!10^n}{3^n}$

c. $\sum_{n=1}^{\infty} \frac{\pi(n+1)^n}{n^{n+1}}$

d. $\sum_{k=1}^{\infty} \frac{k!}{k^k}$

Absolute and Conditional Convergence

Definition (absolute convergent)

The series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

Example

(a) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ converges absolutely because $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series.

(b) Show that the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 \log n}$ is absolutely convergent.

(c) Show that the series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent.

Theorem

If a series converges absolutely, then it converges.

Definition (conditional convergent)

If $\sum_{n=1}^{\infty} a_n$ convergent, but not absolutely convergent, then we say that it is conditionally convergent or that it converges conditionally.

Example

(a) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges conditionally because the series $\sum_{n=1}^{\infty} \left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since it is harmonic series. But we see that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by alternative series test.

(b) Determine whether the following series is absolutely or conditionally convergent, diverges

i. $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{(n+1)\ln(n+1)}$

ii. $\sum_{n=1}^{\infty}\frac{(-1)^n}{(\arctan n)^n}$

Generalized tests

In this test if we take the absolute values on the ratio test and root test the test is said to be generalized ratio test and root test respectively.

a. Generalized ratio test

i. If $\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, then the series $\sum_{n=1}^{\infty}a_n$ absolutely convergent

ii. If $\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ and $\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$, then the series $\sum_{n=1}^{\infty}a_n$ diverges.

iii. If $\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$ then Ratio Test is inconclusive; that is, no conclusion canbe drawn about the convergence or divergence of $\sum_{n=1}^{\infty}a_n$.

Example

Test the following series for absolute convergence

- ✓ $\sum_{n=1}^{\infty}\frac{(-1)^n n^3}{3^n}$
- ✓ $\sum_{n=1}^{\infty}\frac{n^n}{n!}$

Solution

In order to use the generalized ratio test we should observe whether the sequence in the series is positive or negative or both. If the sequence is both negative and alternate in sign it is mandatory to use the generalized ratio test. So accordingly the value of the sequence in the first series is $a_n=\frac{(-1)^n n^3}{3^n}$

As we observe this is alternative series, hence we need to apply generalized ratio test in order to determine convergence.

Thus

$$a_n = \frac{(-1)^n n^3}{3^n}, a_{n+1} = \frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}$$

$$\Rightarrow \rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^3}{3^{n+1}}\times\frac{3^n}{n^3}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^3\times\frac{1}{3}\right| = \frac{1}{3} < 1$$

b. Generalized root test

By this point in your study of series, it may seem as if we have thrown at you a dizzyingarray of different series and tests for convergence or divergence. Just how are you to keep these entire straight? The best suggestion we have is that you work through *many* problems.

We provide a good assortment in the exercise set that follows this section. Some of these require the methods of this section; while others are drawn from the preceding sections (just to keep you thinking about the big picture). For the sake of convenience, this module tries to summarize convergence tests in the table that follows. Enjoy readingcarefully.

Summary of tests

Tests	When to use	Conclusions
Geometric series	$\sum_{n=1}^{\infty} ar^{n-1}$	Converges to $\frac{a}{1-r}$ If $r<1$ And diverges if $r \geq 1$ N.B r is common ratio.
Nth term for divergence	All series	If $\lim_{n\to\infty} a_n \neq 0$, Then the series diverges.

Integral test	$\sum_{n=1}^{\infty} a_n$ where $f(n)=n$ f is continuous and decreasing, $f(x)\geq 0$	$\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)dx$ Both converges or diverges
P-series	$\sum_{n=1}^{\infty}\frac{1}{n^p}$	Converges for $p>1$ And diverges for $p\leq 1$
Comparison test	$\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n, 0\leq a_n\leq b_n$	1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. 2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
Limit Comparison test	$\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n, a_n, b_n > 0, \lim_{n\to\infty}\frac{a_n}{b_n}=L>0$	$\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ Both converge or both diverge.
Alternative series test	$\sum_{n=1}^{\infty}(-1)^{n-1} b_n, (b_n>0, \forall n)$	If the condition ➢ $\lim_{n\to\infty} b_n=0$ ➢ $b_{n+1}<b_n$ ➢ $b_n b_{n+1}<0$, then the series converges.
Absolute convergent	Series with some positive and some negative terms (including alternating series)	If $\sum_{n=1}^{\infty}\lvert a_n\rvert$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

		absolutely. e.g $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$
Conditional convergence	Series with some positive and some negative terms (including alternating series)	If $\sum_{n=1}^{\infty}\lvert a_n\rvert$ diverges, and $\sum_{n=1}^{\infty}a_n$ converges by some tests, then $\sum_{n=1}^{\infty}a_n$ converges conditionally. e.g $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$
Ratio test	Any series (especially those involving exponentials and/or factorials)	For $\rho=\lim_{n\to\infty}a_n$ a. If $0\le\rho<1,$ then $\sum_{n=1}^{\infty}a_n$ converges. b. If $\rho>1,$ then $\sum_{n=1}^{\infty}a_n$ diverges If $\rho=1,$ then no conclusion.
Root test	Any series (especially those involving exponentials)	For $\delta=\lim_{n\to\infty}\sqrt[n]{a_n}$ c. If $0\le\delta<1,$ then $\sum_{n=1}^{\infty}a_n$ converges. d. If $\delta>1,$ then $\sum_{n=1}^{\infty}a_n$ diverges e. If $\delta=1,$ then no conclusion.

(1) Find a formula for the general term of the sequence, considering 1 as the first initial index of the sequence.

(a) $-1,2,7,14,23,---$

(b) $0,2,0,2,0,---$

(c) $1,6,120,5040,362880,---$

(d) $\frac{1}{2\times 3},\frac{2}{3\times 4},\frac{3}{4\times 5},\frac{4}{5\times 6},---$

(e) $1,\frac{-1}{1\times 3},\frac{1}{1\times 3\times 5},\frac{-1}{1\times 3\times 5\times 7},---$

(f) $\frac{1}{\sqrt{\pi}},\frac{4}{\sqrt[3]{\pi}},\frac{9}{\sqrt[4]{\pi}},\frac{16}{\sqrt[5]{\pi}},---$

(g) $(\sqrt{2}-\sqrt{3}),(\sqrt{3}-\sqrt{4}),(\sqrt{4}-\sqrt{5}),---$

(h) $\left(1-\frac{1}{2}\right),\left(\frac{1}{3}-\frac{1}{2}\right),\left(\frac{1}{3}-\frac{1}{4}\right),\left(\frac{1}{5}-\frac{1}{4}\right),---$

(2) Determine whether the following sequences are convergent or divergent.

(a) $a_n=\frac{n}{n+2}$

(b) $a_n=\frac{(n+1)(n+2)}{2n^2}$

(c) $a_n=\left(\frac{n+3}{n+1}\right)^n$

(d) $a_n=\left(1-\frac{2}{n}\right)^n$

(e) $a_n=n\sin\frac{\pi}{n}$

(f) $a_n=\sqrt{n^2+3n}-n$

(g) $a_n=\frac{\ln n}{n}$

(h) $a_n=n\tan^{-1}\left(\frac{1}{n}\right)$

(3) Which of the following sequences are bounded and monotone increasing and decreasing? In all cases consider $n=1$ as the initial index of the sequence.

(a) $a_n=\frac{n}{n+2}$

(b) $a_n=\frac{(n+1)(n+2)}{2n^2}$

(c) $a_n=\left(\frac{n+3}{n+1}\right)^n$

(d) $a_n=\left(1-\frac{2}{n}\right)^n$

(e) $a_n=n\sin\frac{\pi}{n}$

(f) $a_n=\sqrt{n^2+3n}-n$

(g) $a_n=\frac{\ln n}{n}$

(4) Let $\{a_n\}$ be the sequence defined by recursively by

(a) $a_1=\sqrt{2}\ \&\ (a_{n+1})^2=2+a_n, for, n\geq 1$

(b) $a_1 = 1 \,\&\, a_{n+1} = \left[a_n + \frac{3}{a_n}\right], for, n \geq 1$

(c) $a_1 = 1 \,\&\, a_{n+1} = \frac{2+\cos n}{\sqrt{n}} a_n, for, n \geq 1$

Then determine whether the sequence on (a) ,(b),and (c) converges or diverges. If it converges find the limit.

(5) Determine whether the following series are convergent or divergent. Explain the method you used.

(a) $\sum_{n=1}^{\infty} \frac{n2^n (n+1)!}{3^n n!}$

(b) $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(\ln(2n+1))^2}$

(c) $\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$

(d) $\sum_{n=1}^{\infty} \frac{1\times 3\times ---(2n-1)}{(2\times 4\times ---(2n))(3^n+1)}$

(e) $\sum_{n=1}^{\infty} \frac{1}{1+\ln^2 n}$

(f) $\sum_{n=1}^{\infty} \frac{3^{n-1}+1}{3^n}$

(g) $\sum_{n=3}^{\infty} \frac{n!}{n^{n^2}}$

(h) $\sum_{n=3}^{\infty} (-1)^n \frac{20n^2 - n}{n^3 + n^2}$

(i) $\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$

(j) $\sum_{n=1}^{\infty} \frac{2n^2+7n}{3^n(n^2+5n-1)}$

(6) Determine whether the following series are absolutely convergent, or conditionally convergent or either.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}$

(b) $\sum_{n=3}^{\infty} \frac{\cos \pi n}{n\sqrt{n}}$

(c) $\sum_{n=3}^{\infty} \frac{(-1)^n (2n)!}{2^n n! n}$

(d) $\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

(e) $\sum_{n=1}^{\infty} (-1)^n \operatorname{csc} hn$

(f) $\sum_{n=1}^{\infty} (-1)^n \operatorname{sec} hn$

(g) $\sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n^2+1}$

(h) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$

(7) Find the sum of the following series if possible.

(a) $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

(b) $\sum_{n=1}^{\infty}\left(\frac{1}{\ln(n+2)}-\frac{1}{\ln(n+1)}\right)$

(c) $\sum_{n=1}^{\infty}\left(\tan^{-1} n-\tan^{-1}(n+1)\right)$

(d) $\sum_{n=1}^{\infty}\frac{6}{(2n-1)(2n+1)}$

(e) $\sum_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}$

(f) $\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2}$

(g) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$

(h) $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$

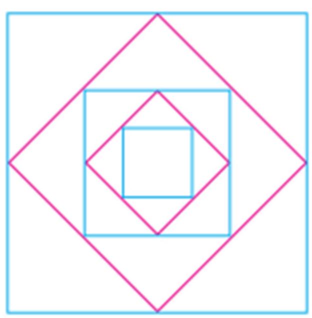
(i) $\frac{1}{1\times 3}+\frac{1}{2\times 4}+\frac{1}{3\times 5}+---$

(j) $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}---$

(8) The accompanying figure shows the first five of a sequence of squares. The outer most square has an area of $4m^2$

Each of the other squares is obtained by joining the midpoints of the sides of the square before it. Find the sum of the area of the sum of

a. five squares (hint: finite series)
b. all squares(hint: infinite series)



(9) A man bought a motor car for Birr 80,000.00. If the value of the car depreciate at the rate of Birr 7000.00 per year. What is its value at the end of the 9th year?

(10) The nth term of a sequence is given by $a_n = 7n-3$

a) Show that the sequence is arithmetic
b) Find the 75th term

(11) A man accepts a position with initial salary of Birr 10470.00 per year. If it is known that his salary will increase at the end of every year by Birr 800.00, what will be his annual salary at the beginning of the 10^{th} year?

(12) If $x, 4x+3 \ \& \ 7x+6$ are consecutive terms of a Geometric sequence. Find the value(s) of x

(13) If $x, 4, y$ are in Geometric sequence and $x, 5, y$ are in arithmetic sequence. Determine the value(s) of $x \ \& \ y$.

(14) The sum of the series $\sum_{n=1}^{\infty} \frac{8}{(n+1)(n+4)}$ is____

Chapter 2

Power series

We now expand our discussion of series to the case where the terms of the series are functions of the variable x. Pay close attention, as the primary reason for studying series is that we can use them to represent functions. This opens up numerous possibilities for us, from approximating the values of transcendental functions to calculating derivatives and integrals of such functions, to studying differential equations. As well, *defining* functions as convergent series produces an explosion of new functions available to us, including many important functions, such as the Bessel functions. We take the first few steps in this section.

As a start, consider the series

$$\sum_{n=0}^{\infty}(x-2)^{n} = 1+(x-2)+(x-2)^2+(x-2)^3+---$$

Notice that for each fixed x, this is a geometric series with $r = x-2$, which will converge whenever $|x-2|<1$ and will diverge when $|x-2|\geq 1$.refer this part from the previous section!

So, in this case we have two things

(a) When the power series converges, that is, $|r|<1$

(b) When the power series diverges, that is, $|r|\geq 1$

- Further, for each x with $|x-2|<1 \Rightarrow -1<x-2<1 \Rightarrow 1<x<3, by, dfn.of, abs, value$,the series converges to $\frac{a}{1-r}=\frac{1}{1-(x-2)}=\frac{1}{3-x}$. That is for each x in the interval $(1,3)$, the power series $\sum_{n=0}^{\infty}(x-2)^{n}=\frac{1}{3-x}$.

N.B: the interval $(1,3)$ is known as the **interval of convergence** of the power series. Bear in mind the big thing that we want to in this chapter is to find the interval of convergence.

- For other values of x with $|x-2|\geq 1 \Rightarrow (-\infty,1]\cup[3,\infty)$, the series diverges.

N.B: the interval $(-\infty,1]\cup[3,\infty)$ is known as the **interval of divergence** of the power series.

In general, any series of the form

Power series

Definition

(1) A power series about $x=0$ is the series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + --- + c_n x^n + ---, eqn1$$

(2) A power series about $x=a$ is the series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + --- + c_n(x-a)^n + ---, eqn2$$

In which the center a and the coefficients $c_0, c_1, c_2, ---, c_n, ---$ are constants.

Example

(a) $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ---$ is a power series centered at zero.

According to the above example we see that this is geometric series with a common ratio $r = x$

Look! The convergence and divergence of a geometric series hugely depend on the value of $r = x$. But this is a variable. So it is convergent if $|r| = |x| < 1$ and its sum is

$$\frac{a}{1-r} = \frac{1}{1=x} = \sum_{n=0}^{\infty} x^n$$

And diverges if $|r| = |x| \geq 1$

Activity 1

a. Find both interval of convergence and divergence of the above power series.
b. Find the coefficients of the above power series.

(b) $1-\frac{(x-2)}{2}+\frac{(x-2)^2}{4}-\frac{(x-2)^3}{8}+---+(-1)^n\frac{(x-2)}{2^n}+---$ is a power series centered at 2 and the coefficients are $c_0=1, c_1=\frac{-1}{2}, c_2=\frac{1}{4}, ---, c_n=\frac{(-1)^n}{2^n}$

Again let we examine whether the series is geometric series or not. Start dividing the second term by the first, the third by second and so on. We see that a series is a geometric series with a common ratio $r=\frac{-(x-2)}{2}$. In the same fashion to above example the series converges for

$|r|=\left|\frac{-(x-2)}{2}\right|=\left|\frac{x-2}{2}\right|<1 \Rightarrow -1<\frac{x-2}{2}<1 \Rightarrow -2<x-2<2 \Rightarrow 0<x<4$ and the sum follows $\frac{a}{1-r}=\frac{1}{1+\frac{(x-2)}{2}}=\frac{2}{2+x-2}=\frac{2}{x}$

Look! How one can write rational function in terms of power series. This will get the answer in the coming lesson till that waits for something bearing in mind this only.

Activity 2

a. Write the above power series in the form summation notation
b. Find interval of convergence and divergence

Determining Where a Power Series Converges

In this part we apply those tests that we apply to test the convergence and divergence of the series in the previous lesson. In this case we use the generalized tests especially ratio test and root test accordingly. This lecture tries to give some examples; so the reader must go beyond these examples.

Examples

(a) Determine the value of the x for which the power series converges

i. $\sum_{n=0}^{\infty}\frac{n}{3^{n+1}}x^n$

ii. $\sum_{n=0}^{\infty}n!x^n$

iii. $\sum_{n=0}^{\infty} \frac{(x-3)^n}{n}$

Solution

By using generalized ratio test we have,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, a_{n+1} = \frac{n+1}{3^{n+2}}x^{n+1}, a_n = \frac{n}{3^{n+1}}x^n$$

$$\Rightarrow \rho = \lim_{n\to\infty}\left|\frac{\frac{n+1}{3^{n+2}}x^{n+1}}{\frac{n}{3^{n+1}}x^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{3^{n+1}*3}x^n * x * \frac{3^{n+1}}{nx^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\right|\left|\frac{x}{3}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\right| * \lim_{n\to\infty}\left|\frac{x}{3}\right| = (1)\left|\frac{x}{3}\right| = \left|\frac{x}{3}\right|$$

So according to the generalized ratio test (ratio test) the series is converges if $\rho < 1$.

Since we are given with that the power series converges and we are required the value of x such that the power series converges.

So we have,

$\rho < 1 \Rightarrow \left|\frac{x}{3}\right| < 1 \Rightarrow -3 < x < 3$,which is a desired.

The series converges absolutely for the value of x in the interval $(-3,3)$ and the series diverges for the value of x in the interval $(-\infty,-3] \cup [3,\infty)$.

Here we face with another problem that whether the series converges or diverges at the end points that is at $x = -3 \,\&\, x = 3$. So we need to test this separately. Wow, learner does not forget to do this for all series that have this kind of question!

i. At $x = 3$,then the series becomes $\sum_{n=0}^{\infty} \frac{n}{3^{n+1}}3^n = \sum_{n=0}^{\infty} \frac{n}{3^n * 3}3^n = \sum_{n=0}^{\infty} \frac{n}{3}$. Again the other question arise: by what test we can test the convergence and divergence of this series? Think over. Ans. by using nth test for divergence we have

$\lim_{n\to\infty} \frac{n}{3} = \infty \neq 0$. So make open interval at this point as it is diverges at this point.

ii. At $x = -3$, then the series becomes

$\sum_{n=0}^{\infty} \frac{n}{3^{n+1}}(-3)^n = \sum_{n=0}^{\infty} \frac{n}{3^n * 3}(-1)^n (3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n}{3}$. Again the other question arise by what test we can test the convergence and divergence of the series? Think over.

The series diverges when $x = -3$ for the same reason.

Hence, the power series converges for all x in the interval $(-3,3)$ and diverges for all x outside of this interval.

Interval and radius of convergence

Notice that the series of the form

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + --- + c_n(x-a)^n + ---$ there is an interval of the form $(a-r, a+r)$ on which the series converges and outside of this interval the series diverges. In the case of the above series $a = 0$. This interval on which a power series converges is called the **interval of convergence.** The constant *r* is called the **radius of convergence** (i.e., *r* is half the lengthof the interval of convergence). As we see in the following result, there is such an interval for every power series.

Theorem

Given any power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, there are exactly three possibilities:

(a) The series converges absolutely for all $x \in (-\infty, \infty)$ and radius of convergence r is ∞.

(b) The series converges only for $x = c$ (diverges for all other values of x) and radius of convergence is $r = 0$

(c) The series converges absolutely for all $x \in (a-r, a+r)$ and diverges for $x < a-r$ and for $x > a+r$ some number r with $0 < r < \infty$

Note that

The radius of convergence r is calculated by the following method.

a. $\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow r = \frac{1}{\rho} \Rightarrow if \rho = 0, then \Rightarrow r = \infty, if \rho = \infty, then \Rightarrow r = 0$

This calculation also works for other tests.

Examples

Finding the Interval and Radius of Convergence

(1) Determine the interval and radius of convergence for the power series

(a) $\sum_{n=0}^{\infty}\frac{10^n(x-1)^n}{n!}$

(b) $\sum_{n=1}^{\infty}\frac{x^n}{n4^n}$

(c) $\sum_{n=0}^{\infty}n!(x-5)^n$

Solution

In all cases we need to find interval and radius of convergences.

(a) By using generalized ratio test we have,

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, a_n = \frac{10^n(x-1)^n}{n!}, a_{n+1} = \frac{10^{n+1}(x-1)^{n+1}}{(n+1)!}$$

$$\Rightarrow \rho = \lim_{n\to\infty}\left|\frac{\frac{10^{n+1}(x-1)^{n+1}}{(n+1)!}}{\frac{10^n(x-1)^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{10^{n+1}(x-1)^{n+1}}{(n+1)!} * \frac{n!}{10^n(x-1)^n}\right|$$

$$\Rightarrow \rho = 10|x-1|\lim_{n\to\infty}\frac{1}{n-1} = 0 < 1, converges, \forall x$$

This says that the series converges absolutely for all x. Thus, the interval ofconvergence for this series is $(-\infty,\infty)$ and the radius of convergence is $r = \frac{1}{\rho} = \infty$.

(b) By using generalized ratio test we have,

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, a_n=\frac{x^n}{n4^n}, a_{n+1}=\frac{x^{n+1}}{(n+1)4^{n+1}}$$

$$\Rightarrow \rho=\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)4^{n+1}}}{\frac{x^n}{n4^n}}\right|=\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)4^{n+1}}*\frac{n4^n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{xn}{4(n+1)}\right|$$

$$\Rightarrow \rho=\left|\frac{x}{4}\right|\lim_{n\to\infty}\frac{n}{n+1}=\frac{|x|}{4}, converges, if, \frac{|x|}{4}<1, diverges, if, \frac{|x|}{4}\geq 1$$

Note that $\rho=\frac{1}{4}\Rightarrow r=4$

So, we are guaranteed absolute convergence for $|x|<4$ and divergence for $|x|>4$. Itremains only to test the endpoints of the interval: $x = \pm 4$. For $x = 4$, we have,

The series $\sum_{n=1}^{\infty}\frac{x^n}{n4^n}$ becomes $\sum_{n=1}^{\infty}\frac{4^n}{n4^n}=\sum_{n=1}^{\infty}\frac{1}{n}$. Now we know that this series diverges as it is harmonic series. So we make an open interval at this point or side, $-,4)$.

For $x=-4$,we have,

The series $\sum_{n=1}^{\infty}\frac{x^n}{n4^n}$ becomes $\sum_{n=1}^{\infty}\frac{(-4)^n}{n4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{n4^n}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$.

In this space, we again know that this series converges by alternative series test (show!)

So we make a closed interval at this point or side, $[-4,-$.

Thus, from the above both intervals the series converges for all x in the half-closed interval $[-4,4)$ and the radius of convergence is $r=4$

(c) By using generalized ratio test we have,

$$\rho=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|, a_n=n!(x-5)^n, a_{n+1}=(n+1)!(x-5)^{n+1}$$

$$\Rightarrow \rho=\lim_{n\to\infty}\left|\frac{(n+1)!(x-5)^{n+1}}{n!(x-5)^n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)n!(x-5)^n(x-5)}{n!(x-5)^n}\right|=\lim_{n\to\infty}\left|\frac{(x-5)(n+1)}{1}\right|$$

$$\Rightarrow \rho=|x-5|\lim_{n\to\infty} n+1=\begin{cases}\infty, if, x\neq 5\\ 0, x=5\end{cases}, converges, only, if, x=5, diverges, \forall x\neq 5$$

Thus this power series converges only if $x=5$ and its radius of convergence is $r=\frac{1}{\rho}=0$.

Differentiating and integrating a power series

Suppose that the power series

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + --- + c_n(x-a)^n + ---$ has radius of convergence $r>0$. Then the series converges absolutely for all $x \in (a-r, a+r)$ and so, defines a function f on the interval $(a-r, a+r)$,

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + --- + c_n(x-a)^n + ---$$

It turns out that such a function is continuous and differentiable.

Definition

(a) of derivative of power series

$$f'(x)=\frac{d}{dx}\sum_{n=0}^{\infty} c_n (x-a)^n = \frac{d}{dx}\left(c_0 + c_1(x-a) + c_2(x-a)^2 + --- + c_n(x-a)^n + ---\right)$$

$$= c_1 + 2c_2(x-a) + --- + nc_n(x-a)^{n-1} + -- = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1},$$

where the radius of convergence of the resulting series is also r. Since we find the derivativeby differentiating each term in the series, we call this **term-by-term** differentiation. Likewise, we can integrate a convergent power series term-by-term,

(b) of integration of power series

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n (x-a)^n dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n (x-a)^{n+1}}{n+1} + K$$

where the radius of convergence of the resulting series is again *r* and where *K* is a constantof integration. The proof of these two results can be found in a text on advanced calculus.

It's important to recognize that these two results are *not* obvious. They are not simply an application of the rule that a derivative or integral of a sum is the sum of the derivatives or integrals, respectively, since a series is not a sum, but rather, a limit of a sum. Further, these results are true for power series, but are *not* true for series in general.

Example (A convergent series whose derivative is divergent)

a. Find the interval of convergence of the series $\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$ and show that the series of derivatives does not converge for any *x*.

Solution

In this example we need to do two things

(1) interval of convergence of the series $\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$

(2) $\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$ Does not converge for any x.

So, let we do them one by one

(1) interval of convergence of the series $\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$

By using comparison test we have,

$\frac{\sin(n^3x)}{n^2} \le \left|\frac{\sin(n^3x)}{n^2}\right| \le \frac{1}{n^2}$. This is because $\left|\sin(n^3x)\right| \le 1$. And we know again that the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges as it is p-series with $p = 2 > 1$. This in turn the series $\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$ converges by comparison test.

(2) The derivative of the series $\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$ is

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2} = \sum_{n=2}^{\infty}\frac{1}{n^2}\frac{d}{dx}\left(\sin(n^3x)\right) = \sum_{n=2}^{\infty}\frac{1}{n^2}n^3\cos(n^3x) = \sum_{n=2}^{\infty}n\cos(n^3x)$$

Again we see that the series $\sum_{n=2}^{\infty}n\cos(n^3x)$ diverges by comparison test **(show this!)**

Therefore, the series $\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin(n^3x)}{n^2}$ does not converge for any x as desired.

Example (differentiating and integrating a power series)

b. Use the power series $\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + ---, |r| = |x| < 1$ to find the power representation of the function $\frac{1}{(1+x)^2}, \frac{1}{1+x^2}$ & $\tan^{-1} x$, $\frac{1}{x+2}, \frac{1}{1+2x}, \frac{3}{2x+3}$

Solution

We notice that the series $\sum_{n=0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + ---, |r| = |x| < 1$ with $r = -x$ is a geometric series which has a sum $\frac{1}{1--x} = \frac{1}{1+x}$

Thus

$\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}, |x| < 1$. So by bearing this in mind we can represent the given function as a power series as the following work.

- We know that derivative of $\frac{1}{1--x} = \frac{1}{1+x}$ is $\frac{-1}{(1+x)^2}$

So,

$$\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = \frac{-1}{(1+x)^2}$$

$$\frac{1}{(1+x)^2} = -\sum_{n=0}^{\infty}(-1)^n \frac{d}{dx}x^n = \sum_{n=0}^{\infty}(-1)^{n+1} nx^{n-1}$$

- And we can represent $\frac{1}{1+x^2}$ by substitution method such that $\sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}$

  So instead of the x in this expression we can write x^2 which follows that

  $$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}, |x| < 1$$

- In order to find the power representation of $\tan^{-1} x$, we know that

  $$\tan^{-1} x = \int \frac{1}{1+x^2} dx$$

  Thus we need to find the integration of the power series that corresponding to $\frac{1}{1+x^2}$. So it follows that,

$$\tan^{-1} x = \int \frac{1}{1+x^2}dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} + K, |x| < 1$$

Multiplication of power series

A converging power series can be multiplied as the way polynomial functions are did.

Theorem

If $\sum_{n=0}^{\infty} a_n x^n = f(x)$ & $\sum_{n=0}^{\infty} b_n x^n = g(x)$ converges absolutely for $|x| < r$, then the product is defined as $c_n = a_o b_n + a_1 b_{n-1} + ---- + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}$ and $\sum_{k=0}^{n} c_n x^n$ converges absolutely to $f(x)g(x)$ for $|x| < r$ and $\left(\sum_{n=0}^{\infty} a_n x^n\right) * \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$

Example

Find the product of the following power series

(a) $\left(\sum_{n=0}^{\infty} x^n\right) * \left(\sum_{n=0}^{\infty} x^n\right)$

(b) $\left(\sum_{n=0}^{\infty} x^n\right) * \left(\sum_{n=0}^{\infty} \frac{1}{2^n}\right)$

Solution

Coming soon

TAYLOR SERIES

Representation of Functions as Power Series

In this section, we develop a compelling reason for considering series. They are not merelya mathematical curiosity, but rather, are an essential means for exploring and computing with transcendental functions ($\sin x, \cos x, \ln x, e^x$)

Suppose that the power series

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + ---- + c_n(x-a)^n + ---$ has radius of convergence $r > 0$. Then the series converges absolutely for all $x \in (a-r, a+r)$ and so, defines a function f on the interval $(a-r, a+r)$,

$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + --- + c_n(x-a)^n + ---$ for each $x \in (a-r, a+r)$ .Differentiating term-by-term, we get that

$$f'(x)=\frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = c_1 + 2c_2(x-a) + --- + nc_n(x-a)^{n-1} + ---- = \sum nc_n(x-a)^{n-1}$$

$$f''(x)=2c_2 + 3.2c_3(x-a) --- + n(n-1)c_n(x-a)^{n-2} + ---- = \sum_{n=0}^{\infty} n(n-1)c_n(x-a)^{n-2}$$

$$f'''(x)=3.2.c_3 + --- + n(n-1)(n-2)c_n x^{n-3} = \sum_{n=0}^{\infty} n(n-1)(n-2)c_n(x-a)^{n-2} \ \& \ so, on$$

All valid for $a-r < x < a+r$ .Notice that if we substitute *x* = *c* in each ofthe above derivatives, all the terms of the series drop out, except one. We get

$f(a)=c_0$
$f'(a)=c_1$
$f''(a)=2c_2$
$f'''(a)=3!c_3$
$f^{(4)}(a)=4!c_4$
.
.
.
$f^{(n)}(x)=n!c_n$

Now solve for c_n we have $c_n = \frac{f^{(n)}(x)}{n!}, n = 1,2,3,---$

To summarize, we found that if $\sum_{n=0}^{\infty} c_n(x-a)^n$ is a convergent power series with radius of Convergence *r* >0, then the series converges to some function *f* that we can write as $f(x)=\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(x-a)^n$ and which is called **Taylor series**. If the center of this power series is zero, then we call this kind of series as **Maclaurin series. And the polynomial is the Taylor polynomial for f expanded about** $x = a$.

There are two important questions we need to answer.

- ❖ Does a series constructed in this way converge and, if so, what is its radius ofconvergence?

❖ If the series converges, it converges to a function. Does it converge to f?

Example

Find the Taylor series expansion and Taylor polynomial for the following function at indicated point whether by using formula or by using Geometric series and find interval of convergence.

(a) $f(x)=e^x, x=0$

(b) $f(x)=\sin x, x=0$

(c) $f(x)=\frac{1}{x}, x=2$

(d) $f(x)=\frac{1}{x+2}, x=1$

(e) $f(x)=\frac{1}{x+1}, x=-1$

(f) $f(x)=\ln(x+1), x=1$

Solution

d.to find the power representation of $f(x)=\frac{1}{x+2}$ at the center 1 that is in the power of $x-1$. So to do this we have,

$$t=x-1 \Rightarrow x=t+1 \Rightarrow f(t+1)=\frac{1}{t+1+2}=\frac{1}{t+3}=\frac{1}{3\left(1+\frac{t}{3}\right)}=\frac{1}{3}\frac{1}{1-\left(-\frac{t}{3}\right)}=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-t}{3}\right)^n, \left|\frac{t}{3}\right|<1$$

$$=\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{t}{3}\right)^n(-1)^n=\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{3^{n+1}}$$

The Taylor polynomial function is $f(x)=\frac{1}{3}-\frac{(x-1)}{3}+---$

Find interval of convergence. (Activity)

For your convenience, we have compiled a list of common Taylor series in the following table.

Taylor Series	Interval of convergence

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty,\infty)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty,\infty)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty,\infty)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\left(x - \frac{\pi}{2}\right)^{2n}$	$(-\infty,\infty)$
$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$	$(0,2]$
$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	(-1.1)

Example (Finding New Taylor Series from Old Ones)

Find the Taylor series in power of x of the following function

a. $e^{2x}, e^{-2x}, e^{\frac{2x}{x^3}}, e^{x^2}$

b. $\ln 3x . \ln(x+1)$

c. $\tan^{-1} 2x, \tan^{-1} x^2$

Solution

a. Rather than compute the Taylor series for these functions from scratch, recall that you that we established before this task.

So accordingly we know that the power representation of $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. So depending on this can answer as following.

In each cases we use substitution.

- So let

$$u = 2x \Rightarrow e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$\Rightarrow e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}, desired, answer$$

- ❖ Let

$$u = x^2 \Rightarrow e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$\Rightarrow e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}, desired, answer$$

Theorem (Binomial series)

Theorem (Fourier series)

This two series will be added soon wait for…….

Summary exercises

1) Given $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}$ Find the interval of convergence of $f(x), f', f'', \int f(x)dx$

2) If the interval of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ is $(0,2)$, then find the interval of convergence of this power series centered at 2.

3) Find radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{(n+p)!x^n}{n!(n+q)!}$,where $p, q \in z^+$

4) Let $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n (n!)^2} = f(x)$,then
   a) Find radius of convergence of f
   b) Show that $xf''(x) + f'(x) + xf(x) = 0, \forall x.$

5) Find the sum of the convergent by using a well-known function. Identify the function and explain how you obtained the sum.

   a) $\sum_{n=0}^{\infty} \frac{1}{2^n n!}$ b) $\sum_{n=0}^{\infty} \frac{2^n}{3^n n!}$ c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n n!}$

6) Find a power series representation of the following function at indicated center (point), and determine the interval of convergence (Note: write at least four non zero terms)

a) $f(x)=\frac{1}{2-x};5$

b) $f(x)=\frac{3}{2x-1};0$

c) $g(x)=\frac{2x}{2x^2+3x-2};0$

d) $f(x)=\frac{-1}{(x+1)^2};0$

e) $f(x)=\frac{1}{(x+1)(x+2)(x+5)};3$

f) $f(x)=\frac{x(x+1)}{(1-x)};0$

g) $f(x)=e^{\sqrt{x}};0$

h) $f(x)=x^2e^x;0$

7) Let $f(x)=x^6-3x^4+2x-1$,then find

   a) The fourth Taylor polynomial of $f(x)$ about -1

   b) The Taylor series of $f(x)$ about -1

8) For what values of x does the power series $\sum_{n=1}^{\infty}\frac{(5-2x)^n}{n}$ absolutely converges?

9) Find the Taylor series Expansion of the following function at the indicated center(point) and determine the interval of convergence (Note: write at least four non zero terms)

   a) $\frac{1}{x+1},2$

   b) $\frac{1}{x+3},2$

   c) $\frac{x+2}{1-x^2},0$

   d) $\sin x,\frac{\pi}{2}$

   e) $\sinh x,\frac{\pi}{2}$

   f) $\ln(2+x),1$

   g) $\tan^{-1}x,0$

   h) $e^x\cos x,0$

   i) $1/(x^2+1),0$

   j) $e^x,1$

   k) $\tan x,0$

   l) $\ln x,2$

10) Find the radius of convergence and interval of convergence of the following power series

   a) $\sum_{n=2}^{\infty}\frac{x^n}{\ln n}$

   b) $\sum_{n=2}^{\infty}\frac{(-1)^n(2n-3)!x^n}{2^n n!}$

   c) $\sum_{n=0}^{\infty}(\sinh 2n)x^n$

   d) $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}x^n}{n(\ln n)^2}$

   e) $\sum_{n=1}^{\infty}\frac{\ln n(x-5)^2}{n+1}$

   f) $\sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n-2)!}$

   g) $\sum_{n=1}^{\infty}\frac{\sin nx^n}{n^2+1}$

   h) $\sum_{n=1}^{\infty}\frac{n!}{n^n}x^n$

i) $\sum_{n=1}^{\infty} \frac{(-1)^n (2x+3)^n}{n \ln n}$

j) $\sum_{n=1}^{\infty} (-1)^n n e^{\frac{-n}{2}} x^{2n}$

11) An electric dipole consists of a charge q at $x = 1$ and $x = -1$. The **electric field** at any $x > 1$ is given by $E(x) = \frac{kq}{(x-1)^2} - \frac{kq}{(x+1)^2}$ for some constant k. Find a power representation of $E(x)$.

12) Let $f(x) = (1+x)^{1/2} + (1-x)^{1/2}$. Find the Maclaurin series for f and use it to find $f^{(4)}(0)$ & $f^{(51)}(0)$.

13) In each case, find the Maclaurin series for f by use of known series and then use it to calculate $f^{(4)}(0)$.

a) $f(x) = e^{x+x^2}$

b) $f(x) = e^{\sin x}$

c) $f(x) = e^{\cos x} = e.e^{\cos x - 1}$

d) $f(x) = \ln(\cos^2 x)$

e) $f(x) = \tan x = \frac{\sin x}{\cos x}$

$f(x) = \int_0^x \frac{e^{t^2} - 1}{t^2} dt$

Chapter 3

Functions of Several Variables

The notation $y = f(x)$ is used to indicate that the variable y depends on the single real variable x, that is, that y is a function of x. The domain of such a function f is a set of real numbers. Many quantities can be regarded as depending on more than one real variable and thus to be functions of more than one variable. For example, the volume of a circular cylinder of radius *r* and height *h* is given $V = \pi r^2 h$; we can say that V is a function of two variables r & h.

If we choose to denote this function by f, then we would write $V = \pi r^2 h$ where

$f(r,h) = \pi r^2 h, (r > 0, h > 0)$

Definition

a. A **function of two variables** is a rule that assigns a real number $f(x,y)$ to each ordered pair of real numbers (x,y) in the domain of the function. For a function f defined on the domain $D \subset R^2$, we sometimes write $f : D \subset R^2 \to R$, to indicate that f maps points in two dimensions to real numbers. You may think of such a function as a rule whose input is a pair of real numbers and whose output is a single real number.
For instance,
    i. $f(x,y) = xy^2$ & $g(x,y) = x^2 - e^y$ are both functions of the two variables *x* and *y*.
b. A **function of three variables** is a rule that assigns a real number $f(x,y,z)$ to each ordered triple of real numbers (x,y,z) in the domain $D \subset R^3$ of the function. We sometimes write $f : D \subset R^3 \to R$ to indicate that *f* maps points in three dimensions to real numbers.
For instance,

ii. $f(x,y,z) = xy^2 \cos z$ & $g(x,y,z) = 3zx^2 - e^y$ are both functions of the three variables *x, y* and *z*.

c. We can similarly define functions of **four (or five or more) variables**, although our focus here is on functions of two and three variables

Examples

Finding the Domain of a Function of Two Variables

1. Find the domain and range of the following function and describe your range and domain by graph

   (a) $f(x,y) = x \ln y$

   (b) $f(x,y) = \sqrt{y - x^2}$

   (c) $f(x,y) = \sin xy$

   (d) $g(x,y) = \dfrac{2x}{y - x^2}$

   (e) $f(x,y,z) = \dfrac{\cos(x+z)}{xy}$

   (f) $f(x,y,z) = \sqrt{9 - x^2 - y^2 - z^2}$

Solution

Function	**Domain**	**Range**
$f(x,y) = x \ln y$	$y > 0$ half-plane lying above x-axis	
$f(x,y) = \sqrt{y - x^2}$	$\{(x,y) : y > x^2\}$	
$f(x,y) = \sin xy$	**Entire plane**	$[-1,1]$
$g(x,y) = \dfrac{2x}{y - x^2}$	$\{(x,y) : y \neq x^2\}$ which is the entire *xy*-plane with the parabola $y = x^2$ removed.	$(-\infty,0) \cup (0,\infty)$

$f(x,y,z)=\dfrac{\cos(x+z)}{xy}$	$\{(x,y,z): x \neq 0 \,\&\, y \neq 0\}$ which is all of three-dimensional space, excluding the *yz*-plane (*x* = 0) and the *xz*-plane (*y* = 0).	
$f(x,y,z)=\sqrt{9-x^2-y^2-z^2}$	$9-x^2-y^2-z^2>0$ the sphere of radius 3 centered at the origin, together with its interior.	
$f(x,y,z)=xy\ln z$	$\{(x,y,z): z>0\}$ Half-space	$(-\infty,\infty)$

Some of the graph of the above functions is:

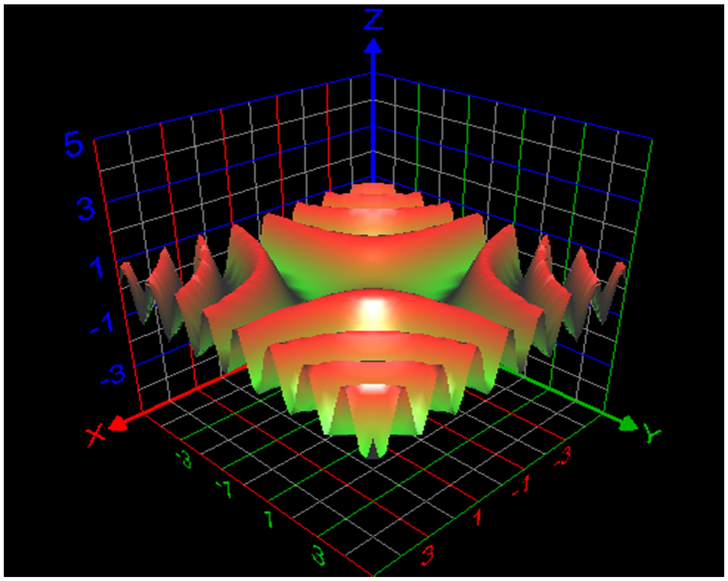


Figure 1 a. $f(x,y)=\sin(xy)$

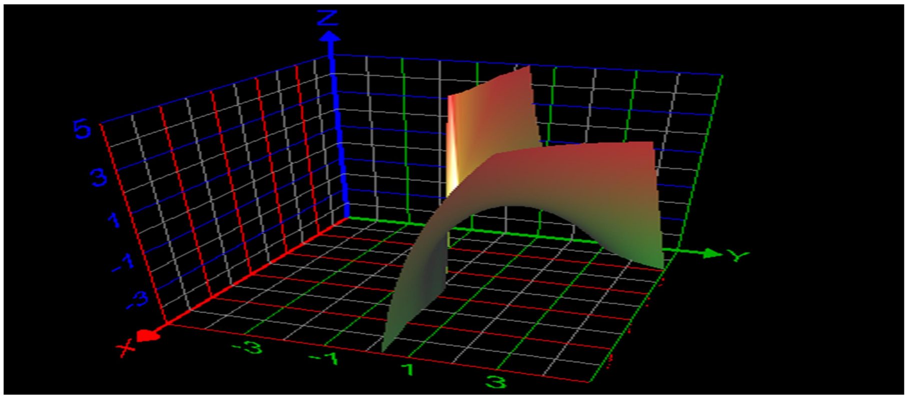


Figure2 the graph of $f(x,y) = x \ln y$

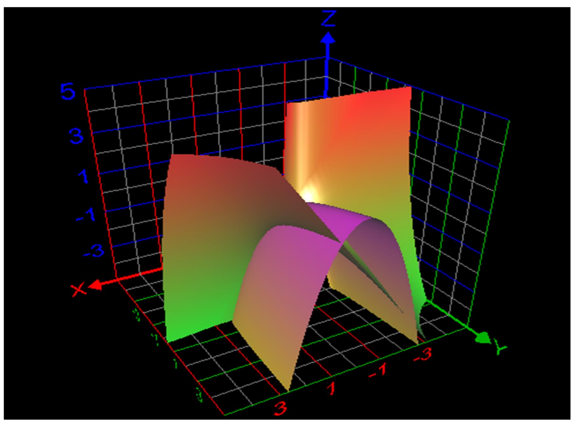


Figure2 the graph of $f(x,y) = \sqrt{y - x^2}$

Definition (level curve, graph, surface)

The set of points in the plane where a function $f(x,y) = c$ has a constant value $f(x,y) = c$ is called a **level curve** of f . The set of all points $(x, y, f(x,y))$ in space for (x,y) in the domain of f , is called the **graph** of f . The graph of f is also called the **surface** $z = f(x,y)$.

Examples

Sketch the graph of the given function and plot the level curves at indicated point. And say something about the level curves. Does it is line, circle, sphere and so on.

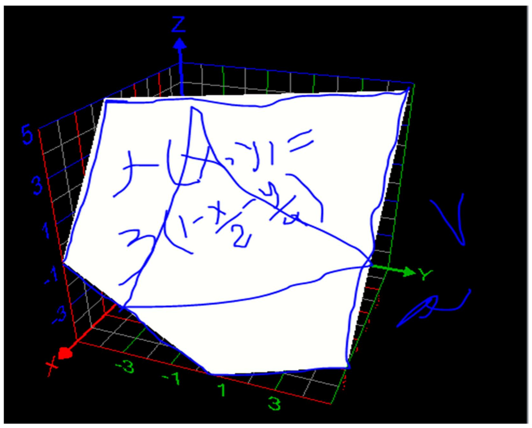
(a) $f(x,y) = 3\left(1 - \frac{x}{2} - \frac{y}{4}\right), (0 \le x \le 2 \;\&\; 0 \le y \le 4 - 2x), c = 0, c = 2$

(b) $f(x,y)=\sqrt{9-x^2-y^2}, c=1, c=2$

(c) $f(x,y)=x^2-y^2, c=1,3$

The graph of the above function seems the following

(a) In order to draw the graph of this graph we let $z=3\left(1-\frac{x}{2}-\frac{y}{4}\right)$. Please one can draw the graph of this by finding the intercepts. So try this! By using mathematical software the graph of this function seems the following.



Limit and continuity

Limit

At the beginning of our study of the calculus and again when we introduced vector-valued functions, we have followed the same progression of topics, beginning with graphs of functions, then limits, continuity, derivatives and integrals. We continue this progression now by extending the concept of limit to functions of two (and then three) variables. As you will see, the increase in dimension causes some interesting complications.

If the values of $f(x,y)$ lie arbitrarily close to a fixed real number *L* for all points (x,y) sufficiently close to a point (x_0,y_0) we say that *f* approaches the limit *L* as (x,y)

approaches (x_0, y_0). This is similar to the informal definition for the limit of a function of a single variable.

Notice, however, that if (x_0, y_0) lies in the interior of $f's$ domain, (x, y) can approach (x_0, y_0) from any direction. The direction of approach can be an **issue**.

Definition

a. Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0).

We will write

$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ if given any number $\in$,we can find a number δ such that $f(x, y)$ satisfies $|f(x, y) - L| < \in$

whenever the distance between, (x,y) and(x_0,y_0) satisfies

$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$

Theorem

Existence of the limit

(a) If $f(x, y) \to L, as (x, y) \to (x_0, y_0), then f(x, y) \to L, as (x, y) \to (x_0, y_0)$ along any smooth curve.

(b) If the limit of $f(x, y)$ fails to exist as $(x, y) \to (x_0, y_0)$ along some smooth curve, or if $f(x, y)$ has different limits as $(x, y) \to (x_0, y_0)$ along two different smooth curves, then the limit of $f(x, y)$ does not exist as $(x, y) \to (x_0, y_0)$.

(c) For the limit to equal *L*, the function has to approach *L* along *every* possible path.

Example

Finding a simple limit

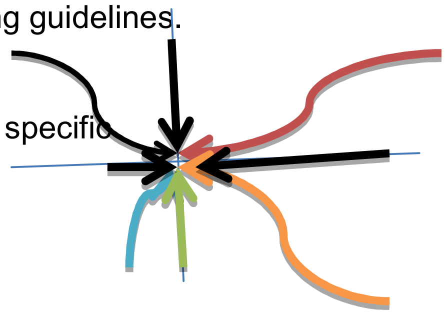
(a) $\lim_{(x,y)\to(1,4)} 5x^3 y^2 - 9 = 5(1)^3 (4)^2 - 9 = 71$

(b) $\lim_{(x,y)\to(2,1)} \dfrac{2x^2 + 3xy}{5xy^2 + 3y} = \dfrac{2 \times 4 + 6}{10 + 3} = \dfrac{14}{13}$

Note in both example we used direct substitution of the limit. The big question wills a raised if the direct substitution fails what method follows to evaluate a limit.

Unlike the case for functions of a single variable where there are just two paths approaching a given point (corresponding to left- and right-hand limits), in two dimensions there are infinitely many paths (and you obviously can't check each one individually).

In practice, when you suspect that a limit does not exist, you should check the limit along the simplest paths first. We will use the following guidelines.



Carefully observe the direction we appraoches to the specific Point from this figure.

The simplist path to try are

(a) Along the line $x = x_{0,} y \to y_0 (vertical, line)$

(b) Along the line $= y_{0,} x \to x_0 (horizontal, line)$

(c) Along the curve $y = g(x), x \to x_0 (where, y_0 = g(x_0))$

(d) Along the curve $x = g(y), y \to y_0 (where, x_0 = g(y_0))$

(e) By converting the rectangular into polar

Properties of limit of function two variables

a.

Example

1. Find the limit of the following function if it exists

(a) $\lim_{(x,y)\to(0.0)} \frac{xy}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0.0)} \frac{xy^2}{x^2+y^4}$

(c) $\lim_{(x,y)\to(0.0)} \frac{5x^2 y}{x^2+y^2}$

(d) $\lim_{(x,y)\to(0.0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$

(e) $\lim_{(x,y)\to(0.0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

(f) $\lim_{(x,y)\to(0.0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

(g) $\lim_{(x,y)\to(0.0)} \left(x^2+y^2\right)\ln(x^2+y^2)$

Solution:

The graph of the function seems the following and to find the limit at the point $(0,0)$ let we use the polar coordinate form by letting (r,θ) be the point of the polar coordinate (x,y) with $r \geq 0$. Then we have $x = r\cos\theta, y = r\sin\theta$. This implies

$x^2+y^2 = r^2$ & $\ln\left(x^2+y^2\right) = \ln\left(r^2\right)$

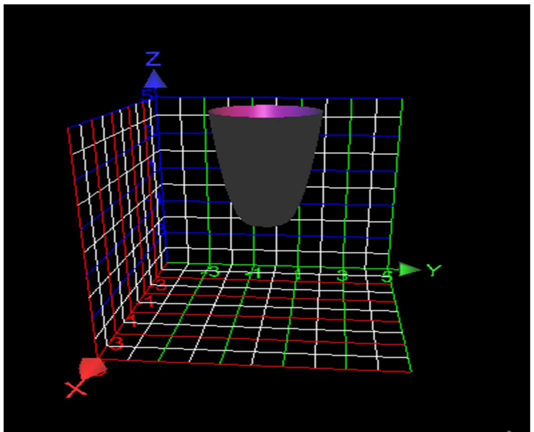
So that $\lim_{(x,y)\to(0.0)} \left(x^2+y^2\right)\ln(x^2+y^2) = \lim_{(r,\theta)\to(0,0)} r^2 \ln\left(r^2\right) = 0.\infty$

This follows that we should apply L'hopital's Rule as $0.\infty$ is one of the kind of indeterminate form and we observe also that it is not in standard form of indeterminate.

$$\lim_{(x,y)\to(0.0)} \left(x^2+y^2\right)\ln(x^2+y^2) = \lim_{r\to 0^+} r^2 \ln\left(r^2\right) = \lim_{r\to 0^+} \frac{\ln\left(r^2\right)}{1/r^2}$$

$$= \lim_{r\to 0^+} \frac{\left(2\ln(r)\right)'}{\left(1/r^2\right)'} = \lim_{r\to 0^+} \frac{2/r}{-2/r^3} = \lim_{r\to 0^+} -r^2 = 0$$

Therefore $\lim_{(x,y)\to(0.0)} (x^2 + y^2)\ln(x^2 + y^2) = 0$



Graph of $f(x,y) = (x^2 + y^2)\ln(x^2 + y^2)$

(h) $\lim_{(x,y,z)\to(0.0,0)} \tan^{-1}\left(\frac{1}{x^2 + y^2 + z^2}\right)$

Definition

The function $f(x,y)$ is continuous at a point (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Activity

1) Can a function $f(x,y) = \frac{\sin x \sin^3 y}{1-\cos(x^2 + y^2)}$ be defined at $(0,0)$ in such a way that it becomes continuous there?

Partial derivatives; tangent lines, higher order partial derivatives

PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

If *z = f(x, y)*, then one can inquire how the value of *z* changes if *y* is held fixed and *x* is allowed to vary, or if *x* is held fixed and *y* is allowed to vary. For example, the ideal gas law in physics states that under appropriate conditions the pressure exerted by a gas is a function of the volume of the gas and its temperature. Thus, a physicist studying gases
might be interested in the rate of change of the pressure if the volume is held

fixed and the temperature is allowed to vary, or if the temperature is held fixed and the volume is allowed to vary. We now define a derivative that describes such rates of change.

Suppose that (x_0, y_0) is a point in the domain of a function $f(x, y)$. If we fix $y = y_0$ then $f(x, y_0)$ is a function of the variable *x* alone. The value of the derivative is derived only with respect to x by fixing y as the constant and in the same manner if we fix $x = x_0$

then $f(x_0, y)$ is a function of the variable y alone then the value of the derivative is derived only with respect to y by fixing x a constant.

Definition (Partial Derivatives)

The first partial derivatives of the function $f(x, y)$ with respect to the variables x & y are the functions $f_x(x, y)$ read as '' $f(x, y)$sub x''or the partial derivative of $f(x, y)$ with respect to x and $f_y(x, y)$ read as '' $f(x, y)$sub y''or the partial derivative of $f(x, y)$ with respect to y and defined as

$f_x(x, y) = \lim_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$ treating y as a constant.........equation 1

$f_y(x, y) = \lim_{h \to 0} \dfrac{f(x, y+h) - f(x, y)}{h}$ treating x as a constant.........equation 2

Notation

Let $z = f(x, y)$. Various notations can be used to denote the partial derivatives of $z = f(x, y)$ considered as functions of x and y:

$\dfrac{\partial}{\partial x} z = \dfrac{\partial}{\partial x} f(x, y) = f_x(x, y) = f_1(x, y) = D_x f(x, y) = D_1 f(x, y)$ And

$\dfrac{\partial}{\partial y} z = \dfrac{\partial}{\partial y} f(x, y) = f_y(x, y) = f_2(x, y) = D_y f(x, y) = D_2 f(x, y)$

Definition

The partial derivative of $f(x, y)$at the point (a, b) are given by

$$f_x(x, y)\big|_{(a,b)} = f_x(a, b) \text{ And}$$

$$f_y(x,y)\big|_{(a,b)} = f_y(a,b)$$

Note $-|$ is called an evaluator symbol

Rule of finding partial derivatives

Let us illustrate the rule by the following examples

Examples

a. Find $f_x(x,y)$ and $f_y(x,y)$ of

   i. $f(x,y) = x^3y^2 + x^4y + y^4$

   ii. $f(x,y) = x^3 + x^2y^3 - 2y^2$

   iii. $f(x,y) = x^{\ln(xy)}$

   iv. $f(x,y) = \log_y^x$

   v. $f(x,y) = \int_y^x \cos(t^2)dt$

   vi. $f(x,y) = (\ln xy)^{\sin\left(\sin^{-1}\left(e^{xy}\right)\right)}$

b. Find $f_x(x,y)$ and $f_y(x,y)$ of $f(x,y) = \dfrac{x+xy}{y+\ln xy}$ at $\left(\dfrac{1}{e}, e\right)$ if it exists.

c. Let $f(x,y) = \begin{cases} \dfrac{-xy}{x^2+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$ Show that the partial derivatives exists at all points (x,y) and discontinuous at $(0,0)$

d. If f is an everywhere differentiable function of one variable, show that $z = f\left(\dfrac{x}{y}\right)$ satisfies the partial differential equation $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial x} = 0$

e. Find $\dfrac{\partial}{\partial x} f\left(x^2y, x+2y\right)$ and $\dfrac{\partial}{\partial y} f\left(x^2y, x+2y\right)$ in terms of partial derivatives of f ,assuming the partial derivatives of f is continuous.

   Solution

- By using a chain rule (single variable)

We have $\frac{\partial z}{\partial x} = f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)$ & $\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)$

Hence,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = x\left(f'\left(\frac{x}{y}\right)\left(\frac{1}{y}\right)\right) + y\left(f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right)\right)$$

$$= \frac{x}{y}f'\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)$$

$$= 0$$

This ends our works.

- We have

$$\frac{\partial}{\partial x}f\left(x^2y, x+2y\right) = \frac{\partial}{\partial x}f\left(x^2y, x+2y\right)\times\frac{\partial}{\partial x}\left(x^2y\right) + \frac{\partial}{\partial y}f\left(x^2y, x+2y\right)\times\frac{\partial}{\partial x}\left(x+2y\right)$$

and

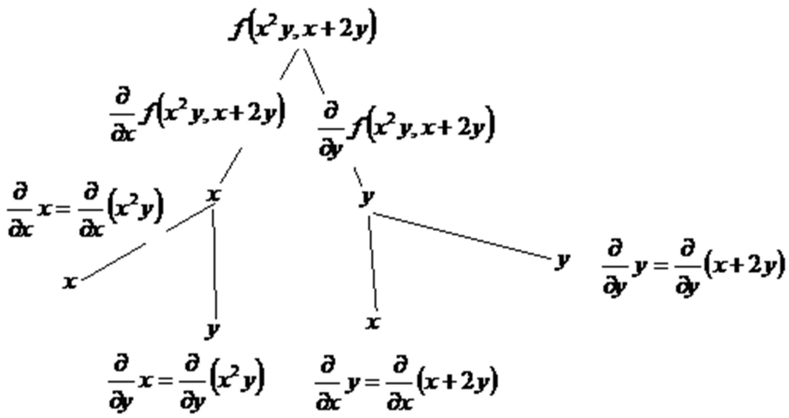
$$= 2xy\frac{\partial}{\partial x}f\left(x^2y, x+2y\right) + \frac{\partial}{\partial y}f\left(x^2y, x+2y\right)$$

- And we have

$$\frac{\partial}{\partial y}f\left(x^2y, x+2y\right) = \frac{\partial}{\partial y}f\left(x^2y, x+2y\right)\times\frac{\partial}{\partial y}\left(x+2y\right) + \frac{\partial}{\partial x}f\left(x^2y, x+2y\right)\times\frac{\partial}{\partial y}\left(x^2y\right)$$

$$= x^2\frac{\partial}{\partial x}f\left(x^2y, x+2y\right) + 2\frac{\partial}{\partial y}f\left(x^2y, x+2y\right)$$

In short this can be recalled by using tree diagram of chain rule



Take the product on each line and sum up which end with the sam independent variables i.e x and y!
note that this is the application of chain rule.

Interpretation of partial derivatives

The partial derivatives $f_x(a,b)$ and $f_y(a,b)$ can be interpreted geometrically as the ***slopes of the tangent lines*** at the point $p(a,b,f(a,b))$ to the traces of the surface and in the planes $y=b$ & $x=a$. Note that $f_x(a,b)$is the slope of the surface $z=f(x,y)$ in x –direction and $f_y(a,b)$is the slope of the surface $z=f(x,y)$in y-direction at the point $p(a,b,f(a,b))$

Note that also the rule that help us to find the partial derivatives of a function of three, four, five, and etc. is the to the rule that we already discussed before in case of a function of two variables. Observe that in case of a function of two and three variables we have two and three first partial derivatives.

Examples

a) Let $f(x,y)=x^2y+5y^3$

Then find the slope of the surface $z=f(x,y)$in both x and y-direction at the point $(1,-2)$

b) The area of parallelogram with adjacent lines $a\,\&\,b$ and included angle θ is given by $area = A = ab\sin\theta$. Find the rate of change A with respect to a and θ where $a = 10, b = 20\ \&\ \theta = \frac{\pi}{6}$

Implicit partial derivatives

Let $z = f(x, y)$ be a given function. And if we want to find the slope of the surface $z = f(x, y)$ in y −direction and x −direction we consider z as a function of both x *and* y. At this case the learner need to refer the definition of implicit differentiation from *Applied Mathematics I* and the detail discussion of this topic will be discussed deeply in the next coming lesson.

Example

1) Find the slope of the sphere $x^2 + y^2 + z^2 = 1$ in y-direction at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.
2) Xzcfsff
3) Cgfxgxh

Higher-order partial derivatives

If f is a function of two variables, then its partial derivatives $f_x\ \&\ f_y$ are also functions of two variables, so we can consider their partial derivatives $f_{xx}, f_{xy}, f_{yx}\ \&\ f_{yy}$, which are called the **second partial derivatives** of f. If $z = f(x, y)$ we use the following

a) $(f_x)_x = \frac{\partial}{\partial x}(f_x) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$ read as d squared f dx squared or f sub xx

b) $(f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$ read as d squared f $dydx$ or f sub xy

c) $(f_y)_x = \frac{\partial}{\partial x}(f_y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$ read as d squared f $dxdy$ or f sub yx

d) $\left(f_y\right)_y = \frac{\partial}{\partial y}\left(f_y\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$ read as d squared f dy squared or f sub yy

Note

i. $\frac{\partial^2 f}{\partial y \partial x}$ Means differentiate f first with respect to x ,then with respect to y

ii. $\frac{\partial^2 f}{\partial x \partial y}$ Means differentiate f first with respect to y ,then with respect to x

iii. If f is continuous and the first partial derivative exists ,then

$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = f_{yx}$ (Euler's Theorem and Clairaut's Theorem)

iv. f_{xy} and f_{yx} are called mixed second order partial derivatives of f (Euler's Theorem and Clairaut's Theorem)

Examples

1. Find all second order partial derivatives of the following function
   a) $f(x, y) = x\cos y + ye^x$
   b) $f(x, y) = y\cos y + \ln(xy)$
   c) $f(x, y) = x\cos^{-1}(\cos(\ln xy)) + ye^x$
   d) $f(x, y) = xy + \frac{e^y}{y^3 + x}$
2. Verify that the function $f(x, y) = xy^2 + ye^{3x}$ is continuous at the point $(0,2)$ by applying Euler's Theorem and Clairaut's Theorem.

Partial derivatives of order 3 or higher can be defined. For instance,

a) $f_{xyy} = \frac{\partial}{\partial y}\left(f_{xy}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}$

b) $f_{xyx} = \frac{\partial}{\partial x}\left(f_{xy}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial x \partial y \partial x}$ And so on.

And using Euler's Theorem and Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous

Example

a. Calculate f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$

b. The equation $u_{xx} + u_{yy} = 0$ and $u_{xx} + u_{yy} + u_{zz} = 0$ is called two-dimension Laplace equation and three-dimension Laplace equation. The solutions to this equation are called harmonic function. Depending on this

   i. Show that the function $u(x, y) = e^x \sin y$ is harmonic function.

   ii. Show that the function $u(x, y, z) = x^2 + y^2 - 2z^2$ is harmonic function

   iii. Show that the function $u(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$ is harmonic function

   iv. Show that the function $u(x, y) = \ln\left(\sqrt{x^2 + y^2}\right)$ is harmonic function

Examples

1. Find all the second order derivatives for

a. $f(x, y) = \cos(2x) - 5xe^y + xy$

b. $f(x, y) = \sin(xy^2)$

c. $f(x, y) = xy^3 - 2x\ln(xy^3)$

Directional derivatives and The Gradient

Directional derivatives

To this point we've only looked at the two partial derivatives $f_y(x, y)$ and $f_x(x, y)$. Recall that these derivatives represent the rate of change of f as we vary *x* (holding *y* fixed) and as we vary *y* (holding *x* fixed) respectively. We now need to discuss how to find the rate of change of f if we allow both *x* and *y* to change simultaneously.

Definition

The rate of change of $f(x, y)$ in the direction of the unit vector $u = ai + bj = \langle a, b \rangle$ is called the directional derivative and is denoted by

$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$

Note specifically, the directional derivative of $f(x,y)$ at a point (x_0, y_0) is given by

$D_u f(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b$

Example

Find the directional derivative of $f(x,y)$ in the direction of indicated vector and at the given point.

a. $f(x,y) = xy, u = \frac{i+j}{\sqrt{2}}(x,y)$

b. $f(x,y) = x^2 + xy, v = i + j, (1,2)$

c. $f(x,y) = 6 - 3x^2 + y^2, v = \frac{i-j}{\sqrt{2}}, (1,2)$

The Gradient

There is another form of the formula that we used to get the directional derivative that is a little nicer and somewhat more compact. It is also a much more general formula that will encompass both of the formulas above.

Definition

If f is a function of two variables x and y, then the **gradient** of $f(x,y)$ is the vector function ∇f defined by

The gradient of $f(x,y) = \nabla f = \text{grad } f(x,y) = f_x(x,y)i + f_y(x,y)j$

Examples

1. Find the gradient of the following at the indicated point.
   a. $f(x,y) = x\cos(xy), (1,-\pi)$
   b. $f(x,y) = xy\cos xy \; , (0,0)$
2. Find the directional derivative of $f(x,y) = xy$ in the direction of gradient at $(1,1)$.

Note we can define a directional derivative and gradient of a given function of three variables in analogous of defining like that of a function of two variables.

Activity

Write a formula of finding a directional derivative and gradient of function of three variables.

Implicit differentiation

Through the chain rule we can more completely describe implicit differentiation which you have studied in applied mathematics I

Definition

Suppose that $f(x,y)$ is differentiable and that the equation $f(x,y)=0$ Fdefines y as a differentiable function x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

Example

1. Let $x^3+y^3=2xy$. Find $\frac{dy}{dx}$

Solution

We can find this $\frac{dy}{dx}$ by either using formula or normal definition of implicit as we did in applied mathematics I.

i. By using formula

To apply this we let $F(x,y)=x^3+y^3-2xy=0$. So by using formula

$\frac{dy}{dx}=-\frac{F_x}{F_y}$ we get $\frac{dy}{dx}=-\frac{(x^3+y^3-2xy)_x}{(x^3+y^3-2xy)_y}=-\frac{3x^2-2y}{3y^2-2x}=\frac{3x^2-2y}{2x-3y^2}$

ii. By using definition

$$\frac{d}{x}\left(x^3 + y^3 = 2xy\right)$$

$$\Rightarrow \frac{dx^3}{dx} + \frac{dy^3}{dx} = \frac{d(2xy)}{dx}$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 2\left(\frac{dx}{dx}.y + x\frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$\Rightarrow 3x^2 - 2y = 2x\frac{dy}{dx} - 3y^2\frac{dy}{dx}$$

$$\Rightarrow 3x^2 - 2y = \left(2x - 3y^2\right)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 2y}{2x - 3y^2}$$

This result is the same to the value we already did on (i)

2. Find $\frac{dy}{dx}$ of the following function

a. $xy + \sin(xy) = y^2$

b. $x\cos(3x) + x^3y^5 = 3x - e^{xy}$

Activity

Find a formula of finding w_x & w_y of a function of three variables $f(x, y, w)$ implicitly

Differentials

Increment and differentials

In this section the concept of Increment and differentials are generalized to a function of two or more variables. Recall that the increment in case of a function of single variable stands for slope of the tangent line which is given by $slope = m = f'(x) = \frac{\Delta y}{\Delta x}$ and for $y = f(x)$ the differential is given by $dy = f'(x)dx$ (go and refer this topic from applied Mathematics I)

Let $z = f(x, y)$ and the increment of x is Δx and the increment of y is Δy. So the increment of z is given by $\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$

Definition

If $z = f(x, y)$ and the increment of x is Δx and the increment of y is Δy, then the differential of the two independent variables $x \,\&\, y$ are

$$dx = \Delta x, dy = \Delta y$$

And the total differential of the dependent variable z is given by

$$dz = f_x dx + f_y dy$$

Note

This definition can be extended to a function more than two. For instance, let $w = f(x, y, z)$ be a function of three variables with three independent variables $x, y \,\&\, z$ and one dependent variable w.

So the increment of $x, y \,\&\, z$ are $\Delta x, \Delta y \,\&\, \Delta z$ respectively and the differential of $x, y \,\&\, z$ are $dx = \Delta x, dy = \Delta y \,\&\, dz = \Delta z$ respectively. The total differential of the dependent variable w is given by $dw = f_x dx + f_y dy + f_z dz$

In the two-variable case, the approximation

$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$ and this can be written as $\Delta f \approx df$ an for $dx = \Delta x, dy = \Delta y$ then we have $\Delta z \approx dz$

Note the point (x_0, y_0) is the initial point such that $\Delta x = x - x_0 \,\&\, \Delta y = y - y_0$

Example

1) Find the total differential of the following function
   d. $z = 2x\sin y - 3x^2y^2$
   e. $w = x^2 + y^2 + z^2$
2) Use the differential dz to approximate the change in $z = f(x, y) = x^2 + 3xy - y^2$ as (x, y) moves from the point (2.3) to the point $(2.05, 2.96)$. Compare this approximation with the actual value in z.

3) Use the differential dz to approximate the change in $z=\sqrt{4-x^2-y^2}$ as (x,y) moves from the point $(1,1)$ to the point $(1.01,0.97)$. Compare this approximation with the actual value or exact change in z.

Solution

We know that $dz = z_x(x_0,y_0)dx + z_y(x_0,y_0)dy \Rightarrow \Delta z = z_x(x_0,y_0)\Delta x + z_y(x_0,y_0)\Delta y$

So we need to find z_x & z_y and x_0 & y_0

$$z_x(x,y) = \frac{-x}{\sqrt{4-x^2-y^2}}$$
$$z_y(x,y) = \frac{-y}{\sqrt{4-x^2-y^2}}$$

And we see that the point is from $(1,1)$ to the point $(1.01,0.97)$ and let we take $(x_0,y_0)=(1,1)$ an initial point and the terminal point

$$(x,y) = (x_0+\Delta x, y_0+\Delta y) = (1.01,0.97) \Rightarrow \Delta x = x - x_0 = 0.01 \,\&\, \Delta y = y - y_0 = -0.03$$

hence, the change in z is approximated by

$$dz = z_x(x_0,y_0)dx + z_y(x_0,y_0)dy$$
$$dz \approx \Delta z \approx \left.\frac{-x}{\sqrt{4-x^2-y^2}}\right|_{(x_0,y_0)=(1,1)} *(0.01) + \left.\frac{-y}{\sqrt{4-x^2-y^2}}\right|_{(x_0,y_0)=(1,1)} *(-0.03)$$
$$\approx 0.0141$$

And the actual change in z is given by

$$\Delta z = \Delta f = f(ter\min al) - f(initial) = f(x,y) - f(x_0,y_0)$$
$$= f(1.01,0.97) - f(1,1)$$
$$\approx 0.0137$$

4) Use differential to approximate $\sqrt{(3.02)^2+(2.98)^2}$

Error analysis

Absolute, relative and percentage change

When we move from the point (x_0, y_0) to nearby point, then change in the value of the function $f(x, y)$ can be described in the following three ways

	True	Estimate
Absolute change	Δf	df
Relative change	$\dfrac{\Delta f}{f(x_0, y_0)}$	$\dfrac{df}{f(x_0, y_0)}$
Percentage change	$\dfrac{\Delta f}{f(x_0, y_0)} \times 100\%$	$\dfrac{df}{f(x_0, y_0)} \times 100\%$

Example

1. The volume $V = \pi r^2 h$ of a right angle circular cylinder is to be calculated from the measured values of $r \,\&\, h$. Suppose that r is measured with the error of no more than 2% and h is measured with the error of no more than 0.5% . Estimate the resulting possible percentage error in calculating V

   Solution

   We are given with $\left|\dfrac{dr}{r} \times 100\right| \leq 2$ and $\left|\dfrac{dh}{h} \times 100\right| \leq 0.5$

   And we need to find percentage error change in volume which is given by

$$\%error = \frac{dV}{V} \times 100\% = \frac{(V_r dr + V_h dh)}{\pi r^2 h} \times 100\% = \frac{(2\pi rhdr + \pi r^2 dh)}{\pi r^2 h} \times 100\%$$

$$= \left(\frac{2dr}{r} + \frac{dh}{h}\right) \times 100\%$$

$$= \left(\frac{2dr}{r} \times 100 + \frac{dh}{h} \times 100\right)\%$$

$$= \left(2\left(\frac{dr}{r} \times 100\right) + \left(\frac{dh}{h} \times 100\right)\right)\%$$

$$= 4 + 0.5$$

$$= 4.5\%$$

Tangent Planes and Normal Lines

If the graph $z = f(x, y)$ is a "smooth" surface near the point P with coordinates $(a, b, f(x, y))$, then that graph will have a tangent plane and a normal line at P. The normal line is the line through P that is perpendicular to the surface; for instance, a line joining a point on a sphere to the centre of the sphere is normal to the sphere. Any nonzero vector that is parallel to the normal line at P is called a normal vector to the surface at P. The tangent plane to the surface $z = f(x, y)$ at P is the plane through P that is perpendicular to the normal line at P.

The tangent plane intersects the vertical plane $y = b$ in a straight line that is tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $y = b$. This line has slope $f_x(a,b)$, so it is parallel to the vector $T_1 = i + f_x(a,b)k$. The tangent plane intersects the vertical plane $x = a$ in a straight line that is tangent at P to the curve of intersection of the surface $z = f(x, y)$ and the plane $x = a$. This line has slope $f_y(a,b)$, so it is parallel to the vector $T_2 = j + f_y(a,b)k$. It follows that the tangent plane, and therefore the surface $z = f(x, y)$ itself, has normal vector

$$n = T_2 \times T_1 = \begin{vmatrix} i & j & k \\ 0 & 1 & f_y(a,b) \\ 1 & 0 & f_x(a,b) \end{vmatrix} = f_x(a,b)i + f_y(a,b)j - k$$

A normal vector to $z = f(x, y)$ at $(a, b, f(a,b))$ is $n = f_x(a,b)i + f_y(a,b)j - k$.

The normal line to $z = f(x, y)$ at $(a, b, f(a,b))$ has the direction vector $f_x(a,b)i + f_y(a,b)j - k$ and has equation

$$\frac{x-a}{f_x(a,b)} = \frac{y-b}{f_y(a,b)} = \frac{z - f(a,b)}{-1}$$

An equation of the tangent plane to $z = f(x, y)$ at $(a, b, f(a,b))$ is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Examples

b. Find a normal vector and equations of the tangent plane and normal line to the graph $z = \sin(xy)$ at the point where $x = \frac{\pi}{3}$ and $y = -1$.Use differential to approximate the equations of the tangent plane when the variable (x, y) vary from the point $(1,2) to (1.3,1.5)$

Chain rule, implicit differentiation

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then f is indirectly a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function. The first version deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t . This means that is indirectly a function of t, $z = f(g(t), h(t))$, and the

Chain Rule gives a formula for differentiating z as a function of t . We assume that f is differentiable. Recall that this is the case when f_x and f_y are continuous.

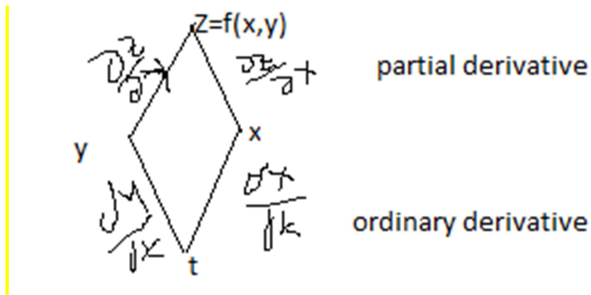
Chain rule Version 1

c. Suppose that $z = f(x, y)$ is a differentiable function of x and y ,where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then $z = f(x, y)$ is a differentiable function of t and

$$\frac{dz}{dt} = f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt} ------eqn(1)$$

$$= z_x(x, y)\frac{dx}{dt} + z_y(x, y)\frac{dy}{dt}$$

This can be recalled by using three diagrams as follows



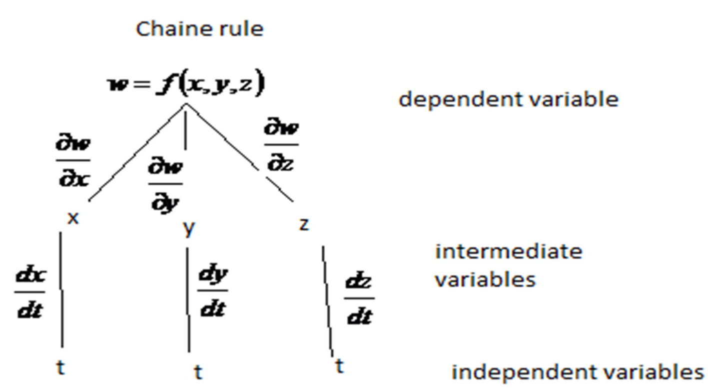
$$\frac{dz}{dt} = \frac{\partial}{\partial x} z(x,y)\frac{dx}{dt} + \frac{\partial}{\partial y} z(x,y)\frac{dy}{dt}$$

Note that the last **independent variable is single** and we have ordinary derivative in case of z ($\frac{dz}{dt}$)

d. Suppose that $w = f(x, y, z)$ is a differentiable function of x , y and z ,where $x = g(t)$, $y = h(t)$ and $z = k(t)$ are differentiable functions of t . Then $w = f(x, y, z)$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x,y,z)\frac{dx}{dt} + f_y(x,y,z)\frac{dy}{dt} + f_z(x,y,z)\frac{dz}{dt} ------ eqn(2)$$
$$= z_x(x,y,z)\frac{dx}{dt} + z_y(x,y,z)\frac{dy}{dt} + w_z(x,y,z)\frac{dz}{dt}$$

This can be recalled by using three diagrams as follows



To read this take the product on each line and add them together. So we get

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial y}\frac{dz}{dt}$$

In the same manner to the above formula we can derive a formula for a function of 4,5 variables and so on

Example

Relative extreme of functions of two variables

Definition

- A function $f(x,y)$ has a relative minimum at the point (a,b)if $f(x,y) \geq f(a,b)$ for all points (x,y)in some region around (a,b).
- A function $f(x,y)$ has a relative maximum at the point (a,b)if $f(x,y) \leq f(a,b)$ for all points (x,y)in some region around (a,b).

Definition

The point (a,b) is a critical point (or a stationary point) of $f(x,y)$ provided that one of the following is true:

a. $\nabla f(a,b) = 0$ (this is equivalent to saying that $f_x(a,b) = 0$ & $f_y(a,b) = 0$)f_x

b. $f_x(a,b) \& / or f_y(a,b)$ does not exist.

Summary Exercises

1. Evaluate the following limits if it exists.

a. $\lim_{(x,y)\to(0,0)} \frac{5x^2 y}{x^4 + y^2}$

b. $\lim_{(x,y)\to(0,0)} \frac{\sin(x - y)}{\cos(x + y)}$

c. $\lim_{(x,y)\to(0,0)} \frac{2x^2 - xy}{4x^2 - y^2}$

2. Let $f(x, y) = xe^{-y} + 5y$. Find the slope of the surface $z = f(x, y)$ in both $x \& y - direction$ at $(3,0)$.

3. Find the indicated partial derivatives of the corresponding function $f_x \& f_y$ of

i. $f(x,y)=\int_{y}^{x} e^{t^2}\,dt$

ii. $f(x,y)=\int_{1}^{xy} e^{t^2}\,dt$

iii. $f(x,y)=\int_{x+y}^{x-y} \sin t^3\,dt$

4. $\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y\partial x}$ & $\frac{\partial^2 z}{\partial y^2}$ of

   i. $z=\sqrt{x}\cos y$

   ii. $z=xy-y^3+x\ln xy$

5. $\left.\frac{\partial^4 u}{\partial x\partial y\partial w\partial z}\right|_{(0,0,1,\pi)}$ of $u(w,x,y,z)=xe^{yw}\sin^2 z$

6. When two resistors having resistances $\Re_1$ ohms and $\Re_2$ ohms are connected in parallel, their combined resistance $\Re_1$ in ohms is $\Re=\frac{\Re_1\Re_2}{\Re_1+\Re_2}$. Show that

   $\frac{\partial^2\Re^2\partial\Re}{\partial^2\Re_1\partial^2\Re_2}=\frac{4\Re^2}{(\Re_1+\Re_2)^4}$.

7. Use a total differential to approximate the change in the values of f from P to Q. Compare your estimate with the actual change in f.

   $f(x,y)=x^2+2xy-4x; P(1,2), Q(1.01,2.04)$

   a. $f(x,y)=\frac{x-y}{xy};-P(1,2),-Q(1.01,2.04)$

   b. $f(x,y,z)=\frac{xyz}{x+y+z}; P(\cancel{1},\cancel{2},4), Q(\cancel{1}.04,\cancel{1}.98,3.97)$

8. Use chain Rule to find the indicated derivatives of the following functions by figuring the appropriate tree diagram for each. $\frac{\partial z}{\partial s},\frac{\partial z}{\partial t}$ of:-

   a) $z=\frac{x}{y}, x=se^{t}, y=1+se^{-t}$

   b) $z=\arctan(2x+y), x=s^2t, y=s\ln t$

9. If $z=f(x,y)$, where f is differentiable

$x = g(t), y = h(t), g(3) = 3, g'(3) = 5, h(3) = 7, h'(3) = 4, f_x(2,7) = 6, and f_y(2,7) = 8$. Then find $\frac{dz}{dt}$ $when t = 3.$

10. Suppose f is differentiable function of $x \,\&\, y, and g(u,v) = f(e^u + \sin v, e^u + \cos v)$ .Use the following table of values to calculate $g_u(0,0) and g_v(0,0)$

	f	g	f_x	f_y
(0,0)	3	6	4	8
(1,2)	6	3	2	5

11. A hiker is standing beside a stream on the side of the mountain, examining her map on the region. The height of the land (in meters) at any point (x, y) is given by the function $f(x, y) = \frac{20{,}000}{3 + x^2 + 2y^2}$

Where $x \,\&\, y$ are measured in kilometer denote the coordinate of the point on hiker's map. The hiker is at point $(3,2)$. Then answer the following depending on the above logic.

   a. What is the direction of flow of the stream at $(3,2)$ on the hiker's map? How fast is the stream descending at her location?
   b. At what angle to the path of the stream (on the map) should the hiker set out if she wishes to climb at a 15° inclination to the horizontal?

12. In what directions at the point $(2,0)$ does the function $f(x, y) = xy$ has the rate of change -1?
13. The temperature $T(x, y)$ at the points of the $xy - plane$ is given by $T(x, y) = x^2 - 2y^2$.

   a. In what direction should an ant a t a position $(2,1)$ if it wishes to cool off as quickly as possible?

b. If the ant moves at the speed of k ,at what rate does it experience the decrease of the temperature?

c. At what rate would the ant experience the decrease of the temperature if it moved from the point $(2,1)$ at speed of k in the direction of the vector $-i-2j.$

d. At what curve through $(2,1)$ should the ant move in order to continue to experience maximum rate of cooling?

14. If $x = e^s \cos t, y = e^s \sin t, and z = v(x,y) = u(x,y).$ Then
    a) Write the appropriate tree diagram of chain Rule Version
    b) show that $\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = (x^2 + y^2)(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2})$

15. let $f(x,y) = \begin{cases} \frac{2xy(x^2-y^2)}{x^2+y^2}, if (x,y) \neq (0,0) \\ 0, if (x,y) = (0,0) \end{cases}$ Then
    i) show that
       a) $f(x,y) = -f(y,x), \forall (x,y)$
       b) $f_x(x,y) = -f_y(y,x)$ and $f_{xy}(x,y) = -f_{yx}(y,x), for (x,y) \neq (0,0)$
       c) $f_x(0,y) = -2y, \forall y$ and ,hence, that $f_{xy}(0,0) = 2$
    ii) Is $f_{xy}(x,y)$ continuous at $(0,0)$?

16. If $f(x,y) = xye^{xy}, \Delta x = -0.1, and \Delta y = 0.2$ Find the total differential of f at $(2,4)$.

17. Find $\frac{dy}{dx}$ implicitly where $F(x,y) = xy + \ln(\sqrt{x+2y+y^2})$

18. Suppose that $D_u = -5 and D_v = 10, where\ u = \frac{3i-4j}{5} and v = \frac{4i+3j}{5}$ find
    a) $f_x(1,2)$ and
    b) $f_y(1,2)$

19. Locate all relative maxima, relative minima, and saddle points, if any of

a) $f(x,y) = y\sin x$

b) $f(x,y) = 12 - 3x - 2y, where \Re; triangular$ region in xy –plane with vertices $(2,0),(0,1),\&(1,2)$

c) $f(x,y) = xy - x^3 - y^2$

d) $f(x,y) = e^{-(x^2+y^2+2x)}$

e) $f(x,y) = \dfrac{2x+2y+1}{x^2+y^2+1}$

f) $f(x,y) = \sin(x+y) + \sin x + \sin y$

g) $f(x,y) = x^3 + y^3 - 18xy$

20. Find critical point of the following function

 a) $f(x,y) = \sqrt{x^2+y^2}$

 b) $f(x,y) = 1-(x+1)^2 + (y-5)^2$

21. Consider the function $f(x,y) = 4x^2 - 3y^2 + 2xy$ over the unit square $0 \le x \le 1, 0 \le y \le 1$. Then

 a) Find the maximum and minimum values of f on each edge of the square

 b) Find the maximum and minimum values of f on each diagonal of the square

 c) Find the maximum and minimum values of f on entire of the square

22. A water line is to be built from point P to point S and must pass through regions where construction costs differ. The cost per kilometers in dollar is 3k from P to Q, 2k from Q to R ,and k from R to$S$. (Hint take $k = 1$). Use Lagrange Multipliers to find $x, y, and\ z$

23. Use Lagrange multipliers to find the extreme value(s) subjected to the given constraint.

 a. $g(x,y) = xy$ subject to the constraint $f(x,y) = x^2 + 2y^2 = 1$

 b. $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 10 = 0$

 c. $f(x,y) = x^3 + y^2$ subject to the constraint $g(x,y) = x^2 + y^2 = 1$

d. $f(x,y) = x^2 + 3y^2 + 2y$ subject to the constraint $h(x,y) = x^2 + y^2 \leq 1$

e. $f(x,y) = x^2 + y^2 - 3x - xy$ subject to the constraint $h(x,y) = x^2 + y^2 \leq 9$

f. $f(x,y,z) = x - y + z$ subject to the constraint $h(x,y,z) = x^2 + y^2 + z^2$